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ABSTRACT

This is part one of a two-part SMSG mathematics text for junior high school students. Key ideas emphasized are structure of arithmetic from an algebraic viewpoint, the real number system as a progressing development, and metric and non-metric relations in geometry. Chapter topics include number line and coordinates, equations, scientific notation, applications of percent, and congruence and the Pythagorean property. Slight revisions are contained in a later edition. (MP)

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# MATHEMATICS FOR JUNIOR HIGH SCHOOL

## VOLUME II (Part 1)

SMSG

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# MATHEMATICS FOR JUNIOR HIGH SCHOOL

## Volume II (Part 1)

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This volume was prepared at a writing session held at the University of Michigan, June 15–August 7, 1959. It is based, in part, on material prepared at the first SMSG writing session, held at Yale University in the summer of 1958. The members of the Ann Arbor writing team were:

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# FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group. This Study Group includes college and university mathematicians, high school teachers of mathematics, experts in education, and representatives of science and technology. The general objective of the Study Group is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by the School Mathematics Study Group was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of high school textbooks which would illustrate such an improved curriculum. This textbook is based upon 14 experimental units which comprised a first product of this project.

The professional mathematicians in the Study Group believe that the mathematics presented in this text is important for all well-educated citizens in our society to know and that it is also important for the pre-college student to learn in preparation for advanced work in the field. At the same time, the high school teachers in the Study Group believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material presented will have a familiar note to it, but the flavor of presentation, the point of view, as it were, will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead a student to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of the commercial textbooks of the future. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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# PREFACE

Fourteen experimental units for use in the seventh and eighth grades were written in the summer of 1958 and tried out by approximately 100 teachers in 12 centers in various parts of the country in the school year 1958-59. Some of the chapters of this book are quite similar to the corresponding experimental units, and in others there are some important changes. Several of the chapters are entirely new. The changes and additions are based both on the teachers' comments, on their experience in teaching the experimental units,and on knowledge of modern needs. The materials are also carefully chosen to provide adequate preparation for the text materials prepared for use in grades 9 through 12 by SMSG.

Big ideas of junior high school mathematics, emphasized in this text are: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; metric and non-metric relations in geometry. Throughout the materials these ideas are associated with their applications. Important at this level are experience with and appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Substantial progress can be made on these concepts in the junior high school.

Mathematics is fascinating to many persons because of its opportunities for creation and discovery as well as for its utility. It is continuously and rapidly growing under the prodding of both intellectual curiosity and practical applications. Even junior high school students may formulate mathematical questions and conjectures which they can test and perhaps settle; they can develop systematic attacks on mathematical problems whether or not the problems have routine or immediately determinable solutions. Recognition of these important factors has played a considerable part in selection of content and method in this text.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that this text may greatly assist all teachers who use it to achieve this highly desirable goal.

# Unit 1

# NUMBER LINE AND COORDINATES

## 1-1. The Number Line

The number line can be very helpful in studying the numbers which we use and in finding out more about their properties. Let us think of the number line as being represented by the line in the drawing. It extends indefinitely far in both directions.

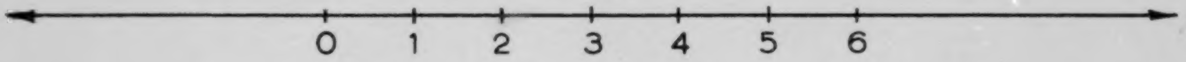


Figure 1-1a

The set of points on the ray to the right of 0 has 0 as the point farthest to the left but there is no point on this ray which is farthest to the right. There are more points on it than there are grains of sand on all the seashores of the world. There are a great many numbers too. Do you think that there are more numbers than there are points on a ray or more points on a ray than there are numbers? Is there a greatest number? If you think of a very large number, you could think of one still greater just by adding 1 to the number you have in mind. Then you could add 1 to that number to get a greater number, and so on and on. All this says that if we start writing the whole numbers

$$0, 1, 2, 3, 4, 5, 6, \ldots$$

we could never finish the task of writing them, even if we lived a long time and had a place where we could write them.

Let us set up a one-to-one correspondence between the whole numbers and some of the points on the number line. We will do it this way. First, select a point on the number line and associate it with the number 0, and label the point 0. Next we select a segment whose measure of length is one unit. This unit may be 1 inch, 1 foot, or just any segment that we choose for our unit length. For example, we may choose the segment P__________Q and use it for the segment whose measure is 1 unit.

Now from the point 0 on the number line measure a distance of 1 unit to the right of 0 to locate the point which we will associate with the number 1. (Although we usually label points with capital letters we will use the number associated with each point to label the point just for the sake of simplicity in the Figure.) Next, from the point labeled 1 measure a distance of 1 unit to the right to locate the point to be associated with the number 2. Label this point with 2. Is it easy to see how the points which are labeled 3, 4, 5, 6 have been located? How would you locate the point which you would label 10?

Can we associate fractions with points on the line? Let us first consider the fractions whose denominators are 2: $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, $\cdots$ . In order to obtain a segment whose measure is $\frac{1}{2}$ we simply get a segment whose length is $\frac{1}{2}$ of the unit segment. Then from the point 0 measure the length of this half-segment to the right of 0. This locates the point with which we associate the number $\frac{1}{2}$ and we label the point $\frac{1}{2}$. The points $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$ are located by repeated measuring along the number line with the segment whose length is $\frac{1}{2}$ unit.

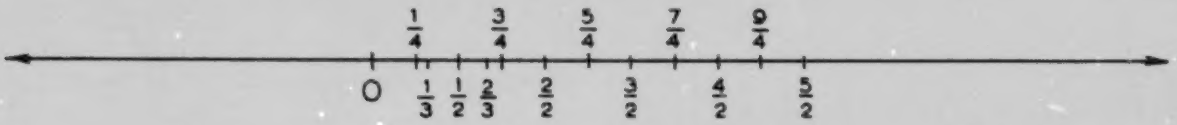


Figure 1-1b

In order to associate points on the number line with fractions whose denominators are 3 we proceed very much as above. Instead of using a length which is $\frac{1}{2}$ of the unit segment we use one which is $\frac{1}{3}$ of the unit segment and measure this length successively to the right of 0. In this way we locate the points $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, .... Similarly, points are located on the number line which are associated with fractions whose denominators are 4, 5, 6, 7.... Some of these are shown in the Figure 1-1b.

Does it appear that we would "use up" all the points on the ray to the right of 0 if we associated each fraction and each counting number with a point on the number line? The answer is that there would still be points not associated with any number--but we will leave further explanation until we get to Chapter V. We have not said anything about associating numbers with points on the ray to the left of 0 and now we should do that.

Before we try to associate numbers with the points on the ray to the left of 0 let us talk about something that is familiar to you, a thermometer scale. We have drawn one in Figure 1-1c. The line on which the scale is indicated is not vertical as the thermometer scale usually is, but it is horizontal so that it will look more like the number line. If the temperature is zero, the end of the column of fluid in the thermometer stem is at 0 on the scale. If the temperature rises the fluid expands and the end of

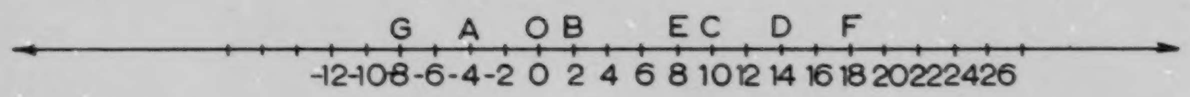


Figure 1-1c

the fluid column moves to the right. When this happens we read the temperature by saying "it is 26 degrees above zero",if the end of the column is at 26 on the scale to the right of 0. If the temperature decreases so that the end of the column is to the left of the point 0 we read the temperature by saying "it is 10 degrees below zero",if the end of the column is at 10 on the scale to the left of 0. The expressions "26 degrees above zero" and "10 degrees below zero" are frequently abbreviated by the use of symbols + and -. These symbols are not used to indicate addition and subtraction but just to indicate directions on the scale from the point 0. Hence we will write $+26^\circ$ and $(-10^\circ)$ to indicate these two temperatures. We will call the + symbol the positive sign and the - symbol the negative sign when they are used in this manner. Hence any temperature reading on the scale to the left of 0 is a negative reading and any one to the right of 0 is a positive reading. We do not say that one is a minus temperature and the other a plus temperature. Thus, $(-10^\circ)$ is read negative ten degrees and it means 10 degrees below zero while $+26^\circ$ is read positive twenty-six degrees and means 26 degrees above zero.

## Class Exercises 1-1a

(All the questions refer to the thermometer scale in Figure 1-1c.)

1. Read and write the temperature if the end of the fluid column is at each of the following points on the scale: A, B, C, O, D, E, F, G.

2. When reading the temperature for the point O on the scale did you need to use either the symbol + or -?

3. What is the temperature if the end of the fluid column is
   - (a) Midway between A and G?
   - (b) Midway between A and C?
   - (c) Midway between B and D?
   - (d) Midway between G and E?
   - (e) Midway between B and F?

4. If the temperature is $0^\circ$ and rises $20^\circ$ what is the temperature then? If it is $0^\circ$ and falls $10^\circ$? If it is $-10^\circ$ and rises $10^\circ$? If is $10^\circ$ and falls $20^\circ$? If it is $20^\circ$ and falls $10^\circ$?

5. Copy the following table and fill in the missing numbers.

| Original Temperature | Temperature Rises | Temperature after the change |
|---|---|---|
| $+10^\circ$ | $10^\circ$ | |
| $-10^\circ$ | $10^\circ$ | |
| | $10^\circ$ | $10^\circ$ |
| | $10^\circ$ | $-20^\circ$ |
| $-40^\circ$ | | $-30^\circ$ |
| $0^\circ$ | | $+20^\circ$ |
| $30^\circ$ | $40^\circ$ | |
| $20^\circ$ | $40^\circ$ | |
| | $20^\circ$ | $-10^\circ$ |
| | $30^\circ$ | $-20^\circ$ |

6. Copy the following table and fill in the missing numbers.

| Original Temperature | Temperature Falls | Temperature after the change |
|---|---|---|
| -20° | 20° | |
| -20° | 10° | |
| -20° | 40° | |
| -10° | 20° | |
| | 30° | 0° |
| | 10° | -10° |
| | 10° | -20° |
| 20° | | -20° |
| 10° | | 0° |

## 1-2. Negative Numbers

In the table in problem 5 you have been adding numbers to other numbers, some of which were positive and some negative. This thermometer scale suggests a way of obtaining numbers which we can associate with points on the number line and also suggests that the numbers which we introduce can be added. Also, the way of writing the numbers on the left ray is suggested by the thermometer scale. But, there are no numbers associated with points on the left ray yet, because we have not introduced any. The thermometer scale just gives us the idea of how these new numbers may be defined. Since the numbers on the left end of the thermometer scale are called negative numbers, we will give this name, negative numbers, to the numbers which we will define and which we will associate with the points on the ray to the left of 0. It seems natural that we would wish to define numbers so that the operations of addition, subtraction, multiplication and division could be carried out with these new numbers just as they are carried out with the numbers on the right ray.

The point associated with 0 is the point which is the intersection point of the right and left rays. This number 0 will be used in defining our new numbers and it will continue to be a very important number.

We will now define, or invent, numbers to be associated with the points on the left ray so that the number 0 will be the additive identity for all the numbers on the line as it is now the additive identity for all rational numbers. Recall that 0 is called the additive identity in the set of rational numbers because

$$0 + a = a + 0 = a, \text{ for any rational number } a.$$

Consider the rational number 2. The new number corresponding to 2 will be written -2, read negative 2, and we define it by saying that it is the number which added to 2 gives 0 as the sum. Hence, $2 + (-2) = 0$. Notice that the definition of (-2) is simply that it is the number which added to 2 gives 0 as the sum. At present there is nothing more that we can say about it. However, we will be able to find out more about it as we go along. We want the commutative property of addition to hold for negative numbers as well as for those we have been using. Hence, we require that $(-2) + 2 = 0$, as well as having $2 + (-2) = 0$ as given by the definition.

Since we will be using the commutative, associative, and distributive properties, let us review them with some examples and exercises. We state them in mathematical language. In the statements a, b, c represent any rational numbers.

Commutative property for addition: $a + b = b + a$.

Associative property for addition: $a + (b + c) = (a + b) + c$

Commutative property for multiplication: $a \cdot b = b \cdot a$

Associative property for multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Distributive property: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

### Exercises 1-2a (Class discussion)

1. State which property (or properties) is used in each of the following:

   (a) $5 - 7 = 7 - 5$

   (b) $3 \cdot (4 \cdot 5) = (3 \cdot 4) \cdot 5$

   (c) $3 \cdot (4 \cdot 5) = (3 \cdot 4) \cdot 5 = (4 \cdot 3) \cdot 5$

   (d) $9 + (1 + 10) = (9 + 1) + 10$

   (e) $9 + (1 + 10) = (1 + 9) + 10$

   (f) $3 \cdot 6 + 3 \cdot 4 = 3 \cdot (6 + 4) = (3 \cdot 10)$

   (g) $7 \cdot 8 + 3 \cdot 8 = (7 + 3) \cdot 8 = (10 \cdot 8)$

   (h) $3 + 17 = 17 + 3$

   (i) $6 \cdot (3 + 7) = (6 \cdot 3) + (6 \cdot 7)$

   (j) $5 \cdot (a + 6) = (5 \cdot a) + (5 \cdot 6)$

Any two numbers whose sum is 0 are called additive inverses of each other if they may be interchanged in the addition. Because of this definition and of what we have said about 2 and -2, these numbers are additive inverses. We can write negative 2 and -2 or (-2) in case there would be any confusion between the sign for the negative number and the symbol for the subtraction operation.

What is the association between this new number, -2, and a point on the number line. Let us agree that -2 will be associated with the point on the number line which is just as far to the left of 0 as the point associated with 2 is to the right of 0. This last statement does not define -2, as it has already been defined by saying that it is the number which added to 2 gives 0 as the sum. This way of locating the point associated with -2 is suggested by the thermometer scale.

Similarly we define negative 3, (-3), as that number which added to 3 gives 0 as the sum. We write

$$3 + (-3) = 0 \text{ and } (-3) + 3 = 0.$$

By definition of additive inverses the numbers 3 and -3 are additive inverses. We associate the number -3 with the point on the number line which is just as far to the left of 0 as the point associated with 3 is to the right of 0. In exactly similar manner to the way in which we defined (or invented) the new numbers (-2) and (-3) we can define new numbers (-1), (-4), $(-\frac{2}{3})$, $(-\frac{5}{6})$ and so on. Indeed, if a is any rational number which is associated with a point on the right half of the number line, then we define negative a, (-a), by saying (-a) is the number which added to a gives 0 as the sum. We write

$$a + (-a) = 0 \text{ and } (-a) + a = 0.$$

The number (-a) is associated with the point on the number line which is the same distance to the left of 0 as the point associated with the number a is to the right of 0. Also, the numbers a and (-a) are additive inverses. Since the use of the statement $a + (-a) = 0$ is a new and somewhat unusual way of stating a definition, it is very important to recognize the statement as a definition. The statements tells us what (-a) is and after that we merely assign a point on the number line to be associated with it.

## Exercises 1-2b (Class Discussion)

1. Define each of the following numbers and tell how you would associate a point on the number line with each one:

$$(-1),\ (-50),\ (-\tfrac{1}{2}),\ (-\tfrac{1}{3}),\ (-100).$$

2. Supply the missing numbers in each of the following statements so that each statement will be a true statement.

(a) $3 + (-3) =$

(b) $\quad + (-4) = 0$

(c) $(-6) + \quad = 0$

(d) $(-75) + (75) =$

(e) $(-\frac{1}{2}) + \quad = 0$

(f) $(\frac{1}{3}) + \quad = 0$

(g) $(-\frac{2}{5}) + \frac{2}{5} =$

(h) $(-.45) + .45 =$

3. Write the additive inverse of each of the following numbers:

$7, (-9), 11, (-12), (-8), 15, (-20), 0, (-\frac{2}{3}), (\frac{4}{9}), (-\frac{7}{8}), (\frac{30}{31})$.

We have called the new numbers negative numbers. The numbers -1, -2, -3, -4, -5,...are called the negative integers. We will now call the counting numbers 1, 2, 3, 4, 5,... the positive integers. Sometimes if we wish to emphasize that a number is positive, we write the + symbol before it. For example, we may write positive 4 as +4 or just 4 without the symbol. If a number is a negative number, the symbol "-" must be written in front of it and we will write the number in parentheses so that the - symbol will not be confused with the subtraction symbol. If there is no possibility of confusion a negative number may be written without the parentheses.

The set of numbers which consists of the positive integers (counting numbers), the negative integers and 0 is called the set of integers. Frequently this set is indicated by the letter I. We write this in concise form:

$$I = \{\ldots.\ -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots.\}$$

in which the numbers appear from left to right just as they would appear if written with the associated points on the number line.

## 1-3. Addition of Positive and Negative Numbers

First, we will consider addition by looking at the example 6 + (-4). In order to obtain this sum we will notice that 4 + (-4) is 0 and write 6 as 2 + 4. Hence 6 + (-4) becomes 2 + 4 + (-4) and since we are going to require that the associative property hold for addition of all numbers we may write

$$6 + (-4) = [2 + 4] + (-4) = 2 + [4 + (-4)]$$

and since $4 + (-4) = 0$, we must have $6 + (-4) = 2$.

We have already stated that the commutative property for addition of all numbers is to hold. As a result of the commutative property, we may write further that

$$6 + (-4) = (-4) + 6 = 2.$$

Example: Find the sum $7 + (-5)$.

We think of 7 as the sum of 2 and 5 and write

$$\begin{aligned} 7 + (-5) &= [2 + 5] + (-5) \\ &= 2 + [5 + (-5)] \\ &= 2 + 0 \\ &= 2. \end{aligned}$$

## Exercises 1-3a

1. Use the method we used in finding $6 + (-4)$ and $7 + (-5)$ to add the following:

| | |
|---|---|
| (a) $9 + (-5)$ | (e) $12 + (-10)$ |
| (b) $10 + (-7)$ | (f) $60 + (-40)$ |
| (c) $8 + (-3)$ | (g) $30 + (-20)$ |
| (d) $6 + (-5)$ | (h) $11 + (-8)$ |

2. Did you notice that all of the parts of Problem 1 were of the type $a + (-b)$ where $a$ and $b$ are positive integers and $a > b$? Use what you observe about the sum, namely that it is $(a - b)$, to obtain the sum in each of the following.

| | |
|---|---|
| (a) $7 + (-4)$ | (e) $15 + (-7)$ |
| (b) $10 + (-7)$ | (f) $15 + (-13)$ |
| (c) $15 + (-7)$ | (g) $25 + (-16)$ |
| (d) $18 + (-10)$ | (h) $30 + (-16)$ |

3. Supply the negative number in each so that each sum will be correct.

   (a) $7 + (\ ) = 2$

   (b) $9 + (\ ) = 6$

   (c) $10 + (\ ) = 4$

   (d) $15 + (\ ) = 10$

   (e) $16 + (\ ) = 6$

   (f) $20 + (\ ) = 9$

In Problem 1 in the above exercises you noticed that in adding $(-b)$ to $a$, the answer was always $a - b$. This is all right if $a > b$. But what can we do if $b > a$? Let us consider this by use of an example: $2 + (-7)$. We would like to think of 2 as being replaced by (some number) + 7. Since $(-5) + 7$ is 2, we will replace $2 + (-7)$ by $[(-5) + 7] + (-7)$ so that we can write

$$\begin{aligned} 2 + (-7) &= [(-5) + 7] + (-7) \\ &= (-5) + [7 + (-7)] \quad \text{Associative property} \\ &= (-5) + 0 \\ &= (-5) \end{aligned}$$

As another example of the type $a + (-b)$ where $b > a$, consider $3 + (-8)$. We will express 3 as $(-5) + 8$ so that we can write

$$\begin{aligned} 3 + (-8) &= [(-5) + 8] + 8 \\ &= (-5) + [(-8) + 8] \\ &= (-5) + 0 \\ &= (-5) \end{aligned}$$

Do you see why we replaced 3 by $(-5) + 8$? We could have replaced 3 by $(-4) + 7$. Why did we not do that? You see that we wished to replace 3 by the sum of two numbers, one of which was 8, so that in the problem the sum of 8 and $(-8)$ would be 0 and so simplify the addition.

## Exercises 1-3b

1. Use the method we have used in finding $3 + (-8)$ and $2 + (-7)$ to find each of the following:

   (a) $5 + (-10)$
   (b) $6 + (-9)$
   (c) $7 + (-12)$
   (d) $3 + (-11)$

2. From the addition problems in No. 1 you see that we have been doing additons of the type $a + (-b)$ where $b > a$. The answers were all of the type $-(b - a)$. Use this short way of writing the answers to each of the following.

   (a) $3 + (-6)$
   (b) $4 + (-9)$
   (c) $6 + (-17)$
   (d) $10 + (-20)$
   (e) $10 + (-30)$
   (f) $15 + (-25)$
   (g) $20 + (-40)$
   (h) $25 + (-50)$

3. Now we "mix them up". Some of these problems are of the type $a + (-b)$ where $a > b$ and in some $b < a$. Find the sum in each.

   (a) $12 + (-10)$
   (b) $(-12) + 10$
   (c) $8 + (-7)$
   (d) $7 + (-8)$
   (e) $11 + (-3)$
   (f) $7 + (-12)$
   (g) $8 + (-15)$
   (h) $7 + (-1)$
   (i) $(-6) + 9$
   (j) $(-6) + 1$

We hope that you have observed the similarity between such problems as $7 + (-3)$ and $3 + (-7)$. We have $7 + (-3) = 4$, but $3 + (-7) = -4$. In general, $a + (-b) = a - b$ if $a > b$ and $a + (b) = -(b - a)$ if $b > a$.

Now we need to know how to add two negative numbers. Consider $(-3) + (-4)$, for example. We know that if we add 3 and $(-3)$

we get 0; and, if we add 4 and (-4) we get 0. Hence [3 + (-3)] + [4 + (-4)] = 0, since the sum in each pair of brackets is 0. We may write

$$[3 + (-3)] + [4 + (-4)] = [3 + 4] + [(-3) + (-4)]$$ by using the associative and commutative properties. But since the sum on the left side of the equality symbol is 0, we must have

$$[3 + 4] + [(-3) + (-4)] = 0,$$

or,

$$7 + [(-3) + (-4)] = 0.$$

This last statement says that the additive inverse of 7 is the number [(-3) + (-4)]. But the additive inverse of 7 is (-7), by definition of additive inverse. Hence, we must conclude that (-3) + (-4) = (-7). Is there an easy way of seeing how these two negative numbers can be added? In general, $(-a) + (-b) = -(a + b)$.

## Exercises 1-3c

1. Find the sum in each.

| | | | |
|---|---|---|---|
| (a) | (-5) + (-4) | (f) | (-25) + (-20) |
| (b) | (-9) + (-1) | (g) | (-17) + (-20) |
| (c) | (-60) + (-4) | (h) | (-24) + (-16) |
| (d) | (-20) + (-10) | (i) | (-15) + (-14) |
| (e) | (-30) + (-20) | (j) | (-25) + (-35) |

## Summary

I. $a + (-b) = a - b$ if a, b are positive rational numbers and $a > b$.

Example: $5 + (-3) = 5 - 3 = 2$

II. $a + (-b) = -(b - a)$ if $a$, $b$ are positive rational numbers and $b > a$.

Example: $7 + (-10) = -(10 - 7) = -3$

III. $(-a) + (-b) = -(a + b)$ if $a$ and $b$ are positive rational numbers.

Example: $(-10) = (-5) = -(10 + 5) = -15$

## 1-4. Multiplication of Negative Numbers

You might have expected to find that subtraction would be considered next, but we are going to leave that until a little later and study multiplication now. We know that any number multiplied by 1 is the number itself if the number is a positive number. The number 1 has this same property by definiton when it is multiplied by a negative number. For this reason the number 1 is called the identity for multiplication. Just as $1 \cdot a = a \cdot 1 = a$, so $1 \cdot (-a) = (-a) \cdot 1 = (-a)$. Let us see if we can show that $(-a) = (-1) \cdot a$. We will start with $a + (-1) \cdot a$ which can be written

$a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a$ where $a$ is any positive rational number. We wish to have the distributive property hold for all numbers. By use of this property we can have

$$\begin{aligned} a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a \\ &= [1 + (-1)] \cdot a \\ &= 0 \cdot a \\ &= 0. \end{aligned}$$

Hence, $a + (-1) \cdot a = 0$. Since this is so, it means that $(-1) \cdot a$ is the additive inverse of $a$. But the additive inverse of $a$ is $(-a)$ and, consequently, we define

$(-a) = (-1) \cdot a$ in order that the distributive property and the property of inverses hold.

Examples: $-3 = -1 \cdot 3$; $-7 = -1 \cdot 7$; $-\frac{2}{3} = -1 \cdot \frac{2}{3}$

We can use the fact that $(-a)$ is equal to $(-1) \cdot a$ to find the product of any two numbers if one of the numbers is positive and one is negative. Consider $(-2) \cdot 3$. By using what we have just shown, we can write

$$\begin{aligned}(-2) \cdot 3 &= [-1 \cdot 2]\, 3 \\ &= -1 \cdot [2 \cdot 3] && \text{by use of the associative property} \\ &= -1 \cdot 6 \\ &= -6.\end{aligned}$$

In general, if $a$ and $b$ are any two positive rational numbers:

$$\begin{aligned}(-a) \cdot b &= [-1 \cdot a] \cdot b \\ &= -1 \cdot [a \cdot b] \\ &= -[ab].\end{aligned}$$

Since the commutative property is to hold for multiplication of negative numbers as well as positive numbers, it follows that

$b \cdot (-a) = (-a) \cdot b = -(a \cdot b) = -(b \cdot a)$

Examples. $6 \cdot (-7) = -(6 \cdot 7) = -42$

$(-5) \cdot 9 = -(5 \cdot 9) = -45$

## Exercises 1-4a

1. Find the product in each.

(a) $9 \cdot (-10) =$

(b) $(-6) \cdot 12 =$

(c) $(-8)(7) =$

(d) $(-\frac{2}{3}) \cdot 3 =$

(e) $(-\frac{7}{8}) \cdot 16 =$

(f) $(-13) \cdot 13 =$

(g) $(-42) \cdot 63 =$

(h) $27 \cdot (-25) =$

(i) $81 \cdot (-\frac{2}{9}) =$

(j) $64 \cdot (-8) =$

(k) $(-2)\cdot(-3)\cdot(4) =$

(l) $(-5)\cdot(-2)\cdot(-3) =$

(m) $(4)\cdot(-5)\cdot(-10) =$

(n) $(2)\cdot(-4)\cdot(-5)\cdot(-1) =$

(o) $(-2)\cdot(-3)\cdot(-1)\cdot(-1) =$

In order to find the product of two negative numbers let us look first at the product of $(-1)$ and $(-1)$.

Since 1 is the identity for multiplication, $-1 = (-1) \cdot 1$. We will need to use this. In order to find $(-1) \cdot (-1)$, let us start by considering the sum

$$(-1) \cdot (-1) + (-1).$$

We now use the fact that $-1 = -1 \cdot 1$ and write

$$\begin{aligned} (-1) \cdot (-1) + (-1) &= (-1) \cdot (-1) + (-1) \cdot 1 \\ &= (-1)[(-1) + 1] \quad \text{Use of the distributive property} \\ &= (-1)\,[0] \quad \text{since } (-1) + 1 = 0 \\ &= 0 \end{aligned}$$

Now if $(-1) \cdot (-1) + (-1) = 0$ this must mean that $(-1)$ is the additive inverse of $(-1) \cdot (-1)$. But $(-1)$ is the additive inverse of 1 and we must conclude that $(-1) \cdot (-1) = 1$.

We can use the fact that $(-1) \cdot (-1) = 1$ in obtaining the product of any two negative numbers. For example, consider the product $(-2) \cdot (-4)$. Using the fact that $(-2) = (-1) \cdot 2$ and $(-4) = (-1) \cdot 4$, we can write

$$
\begin{aligned}
(-2) \cdot (-4) &= (-1) \cdot 2 \cdot (-1) \cdot 4 \\
&= (-1) \cdot (-1) \cdot 2 \cdot 4 \quad \text{Use of commutative and associative properties.} \\
&= 1 \cdot 2 \cdot 4 \\
&= 2 \cdot 4 = 8
\end{aligned}
$$

In general, $(-a)\,(-b) = (-1) \cdot a \cdot (-1) \cdot b = (-1) \cdot (-1) \cdot a \cdot b = a \cdot b$.

## Exercises 1-4b

1. Find the product in each.

    (a) $(-6) \cdot (-10)$
    (b) $(-3) \cdot (-4)$
    (c) $(2\frac{1}{2}) \cdot (6)$
    (d) $(-7\frac{1}{3}) \cdot (-6)$
    (e) $(-5\frac{3}{4}) \cdot (-4)$
    (f) $(-75) \cdot (-4)$
    (g) $(-4) \cdot (-10)$
    (h) $4 \cdot (-10)$
    (i) $(-10) \cdot 4$
    (j) $(-6) \cdot (-7)$
    (k) $(-15) \cdot (-4)$
    (l) $(-20) \cdot (-5\frac{1}{2})$
    (m) $(16) \cdot (-12)$
    (n) $(-16) \cdot (-12)$
    (o) $(-45) \cdot (-3)$
    (p) $25 \cdot (-3)$
    (q) $(-27) \cdot 0$
    (r) $(-16) \cdot (\ 1)$
    (s) $(20) \cdot (-10) \cdot (-5)$
    (t) $(-3) \cdot (-5) \cdot (-4)$
    (u) $(-5) \cdot 6 \cdot (-2)$
    (v) $(-4) \cdot (-5) \cdot 3$
    (w) $(-2) \cdot (-1) \cdot (-3)$
    (x) $(-4) \cdot (-2) \cdot (+2)$
    (y) $(-3) \cdot (-3) \cdot (-3)$
    (z) $(-2) \cdot (2) \cdot (-2)$

2. In the following problems in multiplication put a number in the parentheses so that the statements will be correct.

(a) $(\ )\cdot 6 = -12$

(b) $5\cdot(\ ) = -15$

(c) $(-10)\cdot(\ ) = 100$

(d) $(-5)\cdot(\ ) = 20$

(e) $(-5)\cdot(\ ) = -20$

(f) $11\cdot(\ ) = -110$

(g) $(-1)\cdot(\ ) = 1$

(h) $(-7)\cdot(\ ) = 0$

(i) $1\cdot(\ ) = -1$

(j) $6\cdot(\ ) = -36$

(k) $(-9)\cdot(\ ) = 81$

(l) $5\cdot(\ ) = -30$

(m) $(\ )\cdot(-10) = -90$

(n) $(\ )\cdot(-50) = 100$

(o) $(-6)\cdot(\ ) = -60$

(p) $(-\frac{1}{2})\cdot(\ ) = -1$

## 1-5. Division of Negative Numbers

In Problem 2 of the preceding exercises you were finding a number which could be multiplied by a certain number to yield a given number as the product. For example, you were asked to put the correct number in the parentheses so that

$$5\cdot(\ ) = -15.$$

You obtain the number -3 by recognizing that 5 multiplied by -3 gives -15 as the product. From the meaning of division as the inverse of multiplication $\frac{-15}{5} = -3$. In general, if $a$ and $b$ are any positive rational numbers, and there is a rational number $x$ so that $bx = a$, then the number $x$ is $\frac{a}{b}$, or $a$ divided by $b$.

If $bx = -a$, where $b$ is a positive rational number and $-a$ is a negative rational number, then $x$ must represent a negative number from what we know about multiplication. Hence, $x = \frac{-a}{b}$ is a negative number and we conclude that a negative number divided by a positive number is a negative number.

If $(-b) \cdot x = a$, then $x$ must be a negative number from our knowedge of multiplication. Hence $x = \frac{a}{-b}$ is a negative number and we conclude that a positive number divided by a negative number gives a negative number.

If $(-b) \cdot x = a$, $x$ must be a positive number since $-b$ must be multiplied by a positive number to give $-a$. Hence $x = \frac{-a}{-b}$ is a positive number and we must conclude that a negative number divided by a negative number is a positive number.

These three preceding paragraphs give us the ways of dividing negative numbers. But, there may be a question on how to write such a number as $\frac{-2}{3}$ in which 3 does not divide 2. We may look at it this way: If $3 \cdot x = -2$, then $x = \frac{-2}{3}$ just as if $b \cdot x = a$, then $x = \frac{a}{b}$. Hence, $3 \cdot (\frac{-2}{3}) = -2$. But by multiplication $3 \cdot (-\frac{2}{3}) = -2$ so that $x$ must also be $-\frac{2}{3}$. Thus $\frac{-2}{3} = -\frac{2}{3}$.

Similarly $\frac{2}{-3}$ may be written $-\frac{2}{3}$. For $\frac{2}{-3} = \frac{-1 \cdot 2}{-1 \cdot -3}$ since we may multiply the numerator and denominator of the fraction by $-1$. But, $\frac{-1 \cdot 2}{-1 \cdot -3} = \frac{-2}{3}$ so that $\frac{2}{-3} = \frac{-2}{3}$ or $-\frac{2}{3}$. In general, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$. Example:

$\frac{-7}{3} = \frac{7}{-3} = -\frac{7}{3}$, or if we like we may write $-2\frac{1}{3}$.

## Exercises 1-5

1. Divide in each of the following.

(a) $\frac{-18}{-9}$

(b) $\frac{-25}{5}$

(c) $\frac{-30}{6}$

(d) $\frac{-30}{-6}$

(e) $\frac{30}{-6}$

(f) $\frac{30}{6}$

(g) $\frac{-100}{-20}$

(h) $\frac{-36}{-12}$

(i) $\frac{-432}{12}$

(j) $\frac{-441}{-21}$

(k) $\frac{-484}{22}$

(l) $\frac{-169}{-13}$

(m) $\frac{64}{-16}$

(n) $\frac{750}{-30}$

(o) $\frac{0}{-6}$

(p) $\frac{-39}{-3}$

(q) $\frac{72}{-6}$

(r) $\frac{90}{-15}$

## 1-6. Subtraction of Negative Numbers

We hope you have the addition facts well in mind for we are going to learn subtraction facts for negative numbers by use of addition. This is not a new idea. When you first learned to subtract you probably did it by using addition. For example, in subtracting 2 from 7, we write it 7 - 2 and we recognize that 7 - 2 = 5 by knowing that 2 added to 5 gives 7.

Let us study subtraction by examples first. Then we can state our results in more general terms as we did for addition.

Example 1. Subtract 7 from 2. To get the answer we must ask "What number can be added to 7 to get 2?" That is, 7 + (?) = 2. From our knowledge of addition of negative numbers we recognize that 7 + (-5) = 2. Hence 2 - (7) = (-5). Notice that the answer is (- [7 - 2]) or (-5). Use this method in the next exercises.

### Exercises 1-6a

The symbol "-" is the symbol for subtraction.

1. (a) 2 - (4) =
   (b) 3 - (6) =
   (c) 5 - (7) =
   (d) 4 - (8) =
   (e) 10 - (12) =
   (f) 7 - (9) =

(g) $8 - (10) =$
(h) $7 - (11) =$
(i) $5 - (9) =$
(j) $6 - (8) =$
(k) $4 - (5) =$
(l) $3 - (7) =$
(m) $10 - (5) =$
(n) $12 - (3) =$
(o) $9 - (5) =$
(p) $7 - (3) =$

Now we will subtract a negative number from a positive number.

Example 2. Subtract $(-2)$ from 7. We write this $7 - (-2)$. Again we perform the subtraction by asking for the number which must be added to $(-2)$ to give 7. From our knowledge of addition $(-2) + 9 = 7$. Hence, $7 - (-2) = 9$. We get the same result in subtracting $(-2)$ from 7 that we get in adding (2) to 7.

### Exercises 1-6b

1. Use the method in Example 2 to obtain the answers in the following:

   (a) $6 - (-2) =$
   (b) $9 - (-4) =$
   (c) $11 - (-8) =$
   (d) $10 - (-10) =$
   (e) $6 - (-8) =$
   (f) $6 - (-6) =$
   (g) $7 - (-9) =$
   (h) $7 - (-5) =$
   (i) $20 - (-20) =$
   (j) $15 - (-10) =$
   (k) $30 - (-20) =$
   (l) $40 - (-40) =$

For the third type of subtraction we subtract a positive number from a negative number.

Example 3. Subtract 7 from $(-2)$. We write this $(-2) - (7)$ and ask for the number which must be added to 7 to give $(-2)$. We know that $7 + (-9) = (-2)$ so we know that $(-2) - (7) = (-9)$. Does this show that subtracting 7 from $(-2)$ is the same as adding $(-7)$

to (-2)?

## Exercises 1-6c

1. Perform each of the following subtractions.

    (a) (-3) - (7) =

    (b) (-2) - (9) =

    (c) (-5) - (10) =

    (d) (-8) - (8) =

    (e) (-6) - (9) =

    (f) (-7) - (10) =

    (g) (-12) - (11) =

    (h) (-15) - (10) =

    (i) (-4) - (2) =

    (j) (-5) - (3) =

    (k) (-7) - (7) =

    (l) (-6) - (6) =

The fourth, and last, type of subtraction is the subtraction of one negative number from another negative number. We consider this in the next example.

Example 4. Subtract (-7) from (-2). We write (-2) - (-7) and ask for the number which must be added to (-7) to give (-2). Since (-7) + (5) = -2 we have (-2) - (-7) = 5.

Is (-7) - (-2) equal to (-2) - (-7)? We can tell by finding what we must add to (-2) to get (-7). Since (-5) must be added to (-2) to get (-7), we have (-7) - (-2) = (-5), but (-2) - (-7) = 5.

Do you see that (-2) - (-7) = 5 is the same as (-2) + 7 = 5, and (-7) - (-2) = (-5) is the same as (-7) + 2 = (-5)?

## Exercises 1-6d

1. Perform the indicated subtraction in each of the following.

    Example: (-5) - (-3) = . To get the answer ask yourself what number must be added to (-3) to get (-5). Since (-3) + (-2) = -5, (-5) - (-3) = (-2).

(a) (-4) - (-2) =

(b) (-6) - (-5) =

(c) (-2) - (-1) =

(d) (-5) - (-2) =

(e) (-2) - (-4) =

(f) (-3) - (-5) =

(g) (-4) - (-2) =

(h) (-5) - (-1) =

(i) (-1) - (-5) =

(j) (-4) - (-1) =

(k) (-6) - (-4) =

(l) (-4) - (-6) =

Summary of Subtraction of Negative Numbers. In the summary a and b represent positive numbers, (-a) and (-b) represent negative numbers. Whenever the "-" symbol is used within parentheses, it is the negative sign for the number with which it is written; when used in any other way it is the symbol for subtraction.

| | General Statement | Example |
|---|---|---|
| 1. | a - b = a + (-b) = -(b - a), if b > a | 7 - 9 = 7 + (-9) = (-2) |
| 2. | a - (-b) = a + b | 5 - (-3) = 5 + 3 = 8 |
| 3. | (-a) - b = - (a + b) | (-10) - 8 = -(10 + 8) = -18 |
| 4. | (-a) - (-b) = (-a) + b | (-2) - (-7) = (-2) + 7 = 5 |

### Exercises 1-6e

Now let us use all of these types of examples.

1. Perform the following subtractions.

(a) (-10) - (-3) =

(b) 4 - 6 =

(c) 16 - 12 =

(d) 8 - (-2) =

(e) (-8) - 2 =

(f) (-8) - (-2) =

(g) (-9) - 2 =

(h) 9 - (-3) =

(i) 7 - (5) =

(j) 7 - (-5) =

(k) 2 - 9 =

(l) 2 - (-9) =

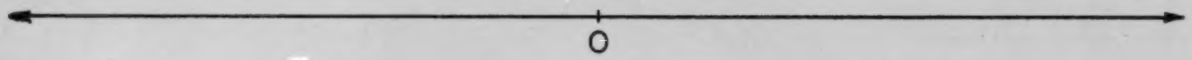
(m) 3 - 10 =

(n) 3 - (-10) =

(o) 4 - (-7) =

## 1-7. Coordinates on the Line

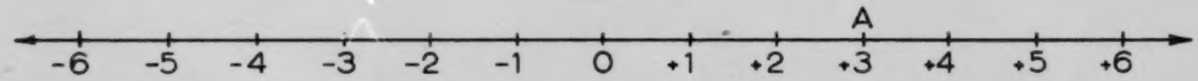
Let us consider the number line.



Choose a point on the line and label it 0. This point we call the origin.

From Section 1-1 recall that points on the ray to the right of 0 (the origin) will be associated with positive numbers and points to the left of the origin will be associated with negative numbers. We are now able to locate a point on the line associated with any rational number.

To illustrate the location of points associated with several rational numbers on a number line, let us draw a line segment 6 inches in length.



In labeling our number line it is convenient to use only integers.

To locate point A on the number line we say it is three units from the origin on the ray to the right of the origin. The distance of a point from the origin with its direction indicated by either + or - is called the coordinate of the point.

A rational number can always be associated with a point on the number line. The rational number gives the number of units distance of the point from 0. If the sign of the rational number is positive, the point is to the right of 0; and if negative, the point is to the left of 0. Point A may be written A(+3) where its coordinate is (+3).

Locate point B(-2) and point C(0) on the number line. Could you locate point $D(+\frac{3}{2})$ and point $E(+\frac{5}{2})$?

By the method used in these above examples we can set up a one-to-one correspondence between the set of positive and negative rational numbers and a set of points on the number line. With any one of these numbers there is associated a point on the line. The number that is associated with a point on the line is called its coordinate. The coordinate of A(+3) is (+3); the coordinate of B(-2) is (-2).

## Exercises 1-7

1. Draw a segment of a number line 5 inches in length. Let one unit be one inch. On the line locate the following points.

   $A(-1)$, $B(+\frac{5}{2})$, $C(1)$, $T(0)$, $L(-\frac{3}{2})$, $P(-2)$

2. In problem 1, how far is it in inches between the point labeled T and the point labeled L? between P and B? between L and B? from the origin to A?

3. Using a number line with 1 inch as the unit of length, mark the following points.

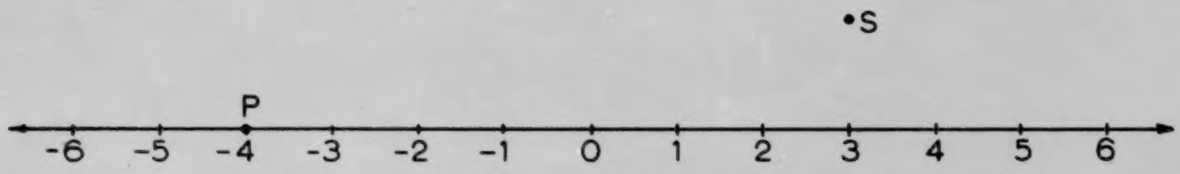
   $R(\frac{1}{3})$, $S(\frac{5}{6})$, $D(-\frac{3}{2})$, $F(0)$, $E(+\frac{3}{2})$

4. If the line segment in problem 3 were a highway and was drawn to scale where 1 inch represents 1 mile, how far in miles is it between these points on the highway: F and R? D and E?

5. If your number line were vertical instead of horizontal and your number scale were written with positive integers above the origin and negative numbers below the origin, draw a number line and label points to correspond with the rational numbers, 0, 1, 2, 3, -1, -2, -3, -4.

## 1-8. Coordinate System in Plane

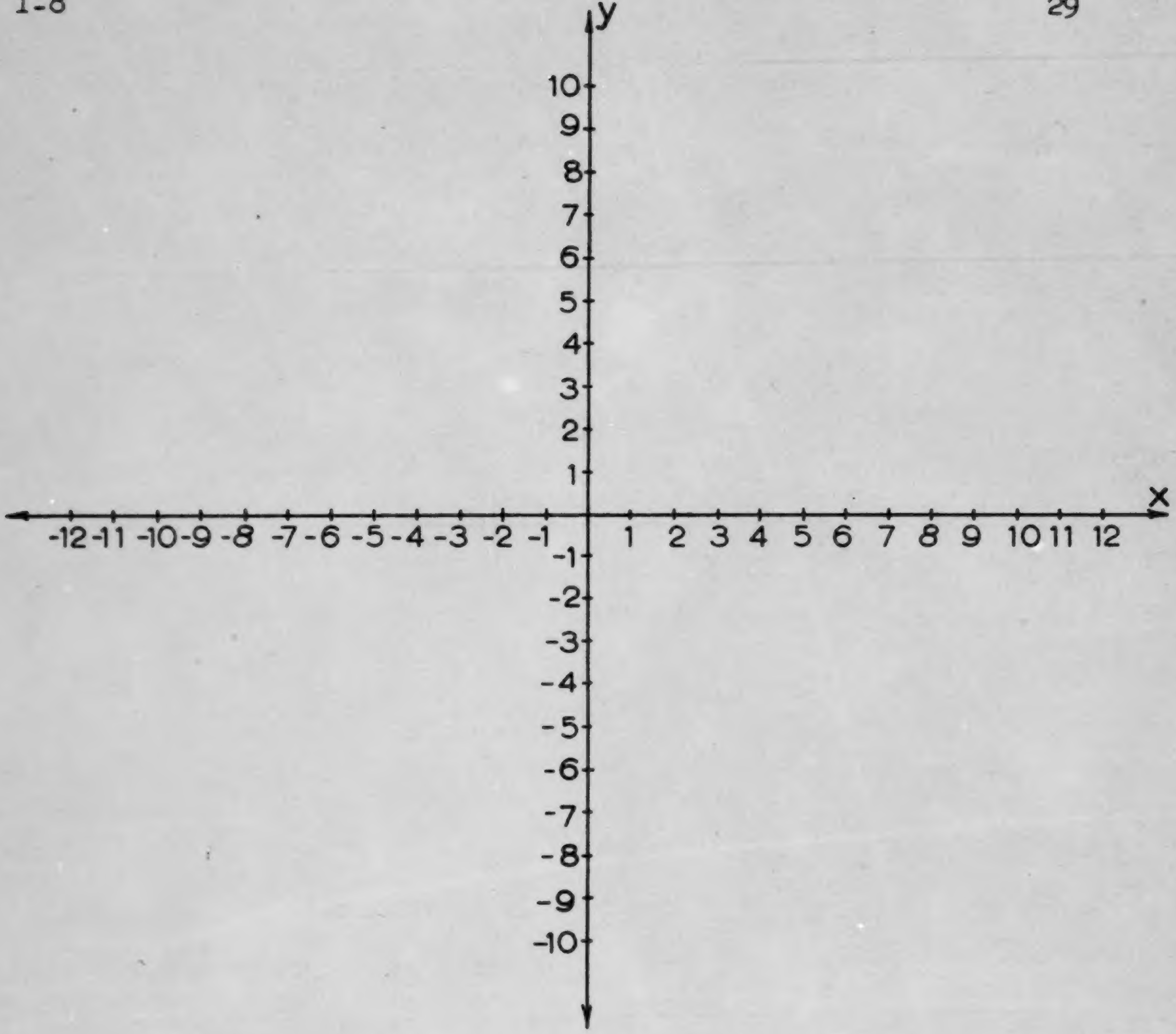
You have learned that a single coordinate can locate a point on a number line. Point P(-4) is 4 units to the left of the origin. Does the method for locating a point on the line give you a way of telling how to locate the point if it is not on the number line?



Point S is not on the number line. Could you say that the coordinate of S is (+3)? Why not? Find the point on the number line with coordinate (+3). Is it the same point as point S?

How many number coordinates determine point P on the number line? Will one number coordinate determine point S? No, you need more than one number to determine point S on the half plane above the number line.

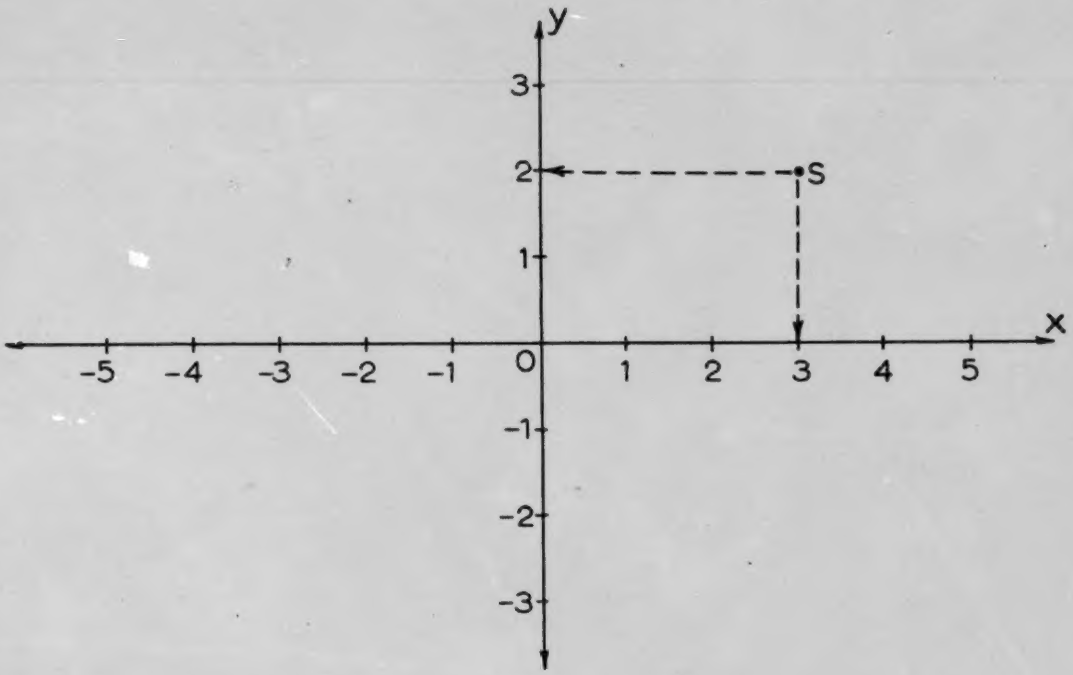
To help locate point S, draw a vertical number line perpendicular to the horizontal number and intersecting it at the origin. Use the same unit of measure for your scale on the vertical number line as you used on the horizontal number line. Label both scales.


y
x
10
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8 9 10 11 12

Let us agree to call the horizontal line the X-axis and the vertical line the Y-axis to simplify talking about them. When we refer to both the X-axis and Y-axis, we will call them the axes (plural of axis).

Now we can be a little more precise in locating point S. It lies in the half plane above the X-axis and in the half plane to the right of the Y-axis.

To help us determine the coordinates of point S, draw a line segment from point S perpendicular to the X-axis. The number associated with the point where it intersects the X-axis is called the x-coordinate of S.

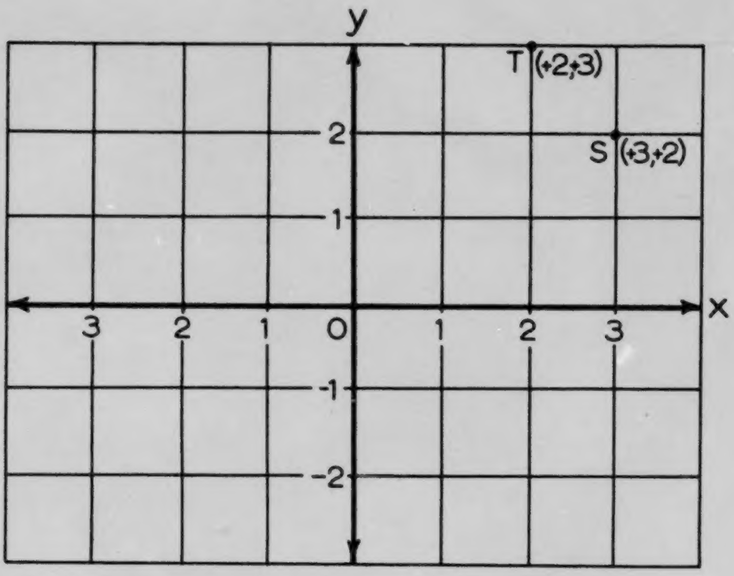


Now draw a line perpendicular to the Y-axis from point S. The number associated with the point where it intersects the Y-axis is called the y-coordinate of S. This system of coordinates is called a rectangular system of coordinates. Can you see why?

Point S has an x-coordinate of (+3) and a y-coordinate of (+2), which we write (+3, +2) with the x-coordinate always written before the y-coordinate.

Each point in the plane has a pair of numbers associated with it. The first number of the pair gives the x-coordinate of the point, the second number gives the y-coordinate of the point. These numbers determine its position in the plane.

On a piece of squared paper draw the X-axis and the Y-axis. Choose a convenient length for 1 unit and use the same length to mark off a scale on both the axes.



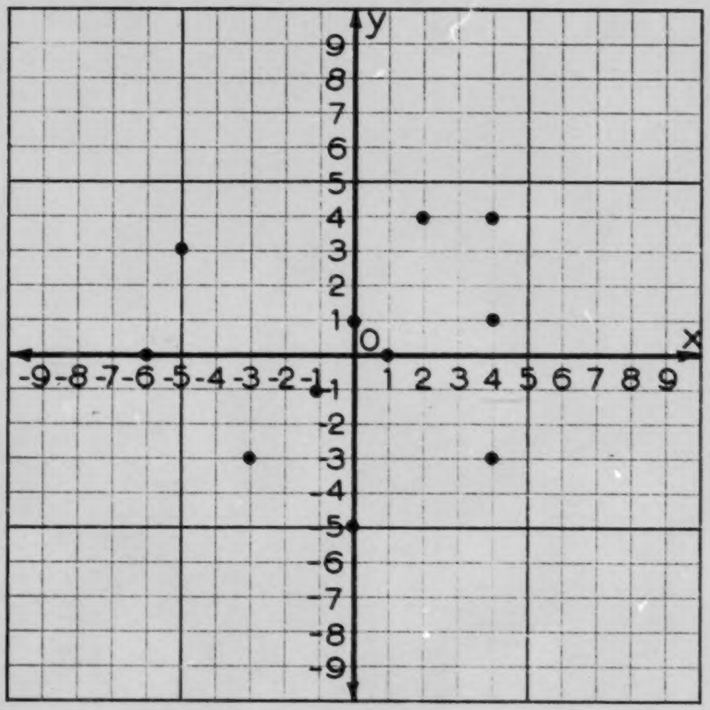
Mark point (+3, +2) on the half plane above the X-axis. Is this the same point (+2, +3)? How does it differ from (+3, +2)? By interchanging the numbers in the number pair (3, 2), we get the number pair (2, 3). But the number pair (3, 2) is associated with the point S and (2, 3) is associated with the point T.

Suppose you were giving a friend directions to a certain place in a city laid out in rectangular blocks. If you told him to go 3 blocks east and 2 blocks north, would this be the same as telling him to go 2 blocks east and 3 blocks north? Of course not. The numbers are the same, but the order in which they are given is quite different. The first set of instructions might be written (3 east, 2 north) and the second set (2 east, 3 north). As you can easily see, these instructions lead to two completely different locations. Do you see why it is important to watch the order of a pair of number coordinates?

If we interchange the numbers in a number pair, will the points associated with the number pair always be different points? Plot the point (-1, 4). Label it. Now interchange this pair of numbers to get (4, -1). Plot this point. Is it different from (-1, 4)? Can you think of any exceptions to this? How about the number pair (2, 2)? List a few ordered pairs which will give you the same point if their order is reversed.

Given the following ordered pairs of numbers, locate the points in the plane associated with these pairs.

(4, 1), (1, 0), (0, 1), (2, 4), (4, 4), (-1, -1), (-3, 3), (4, -3), (-5, 3), (0, -5), (-6, 0). Write the ordered pair of numbers beside the point. Locating and marking the point with respect to the X-axis and the Y-axis is called plotting the points.



## Exercises 1-8a

1. On squared paper draw a pair of axes and label them. Plot the following points. To the right of each point place the coordinates in proper notation. Use the same pair of axes for all examples.

(a) (6, -3)
(-7, -1)
(-9, -7)
(5, -1)
(-8, 10)
(0, 0)
(-1, -1)
(4, 3)

(b) (1, 1)
(6, -5)
(-3, -3)
(4, -10)
(-9, -6)
(-8, 0)
(0, -5)
(-2, -5)

2. What is the y-coordinate of any point on the X-axis?

3. What is the x-coordinate of any point on the Y-axis?

4. What are the coordinates of the origin?

5. Plot the points (-1, -1), (8, -1), (-1, 3) and (8, 3). Show that these points are the vertices of a rectangle.

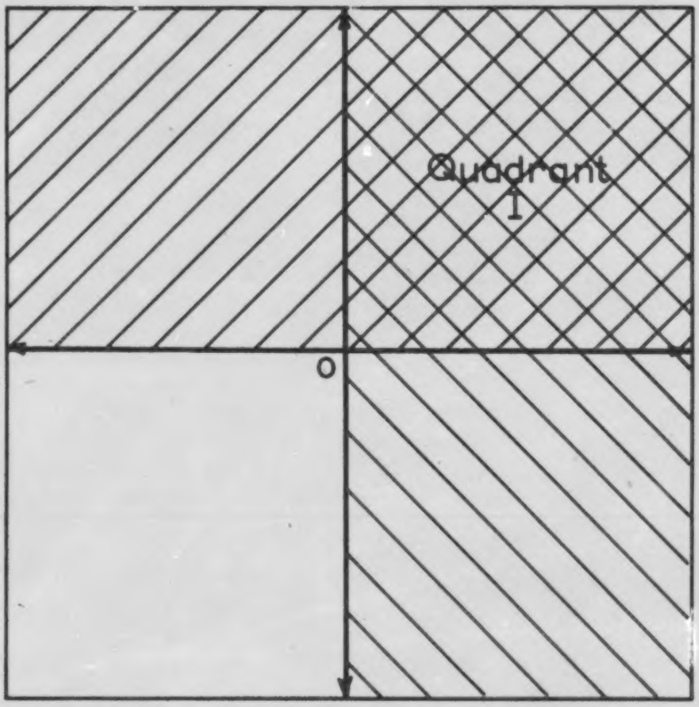
6. Plot the points (3, 6), (3, -6), (-3, 6) and (-3, -6). What kind of figure is evident if these are the vertices?

7. Three vertices of a rectangle are: (0, 4), (7, 4) and (0, 0). Find the coordinates of the 4th vertex.

8. Plot the following points:
(0, 5), (3, 4), (4, 3), (5, 0), (4, -3), (3, -4), (0, -5), (-3, -4), (-4, -3), (-5, 0), (-4, 3), (-3, 4).
Do these points appear to be on a circle? What is the center? What is the length of the radius?

## Quadrants

Did you notice that the half planes above and below the X-axis intersect the half planes to the right and to the left of the Y-axis? These intersections are called quadrants and are numbered counter-clockwise with Quadrant I being the intersection of the half plane above the X-axis and the half plane to the right of the Y-axis.



Points in the intersection set of these two half planes are in the first quadrant or Quadrant I. The intersection of the half plane above the X-axis and half plane to the left of the Y-axis is Quadrant II. Quadrant III is the intersection of the half plane below the X-axis and to the left of the Y-axis. Quadrant IV is the intersection of the half plane to the right of the

Y-axis and the half plane below the X-axis.

The numbers in order pairs may be positive, negative, or zero as you have noticed in the exercises. Both numbers of the pair may be positive, both numbers may be negative, one may be positive, and one may be negative, one may be zero or both may be zero.

Where are all points for which both numbers in the ordered pair are positive? Will they be in the same quadrant? How can you tell?

Where are all the points for which both numbers of the ordered pair are negative? Show this by plotting some points. Can you predict with 100% accuracy in which quadrant the point lies if you know its x and y-coordinates?

What can you tell about the point (-4, 3)? In which quadrant is it? Is (3, -4) in the same quadrant? Why not?

## Exercises 1-8b

1. Given the following ordered pairs of numbers: Write the number of the quadrant in which you find the point representing each of these ordered pairs.

| Ordered Pair | Quadrant |
|---|---|
| (3, 5) | ________ |
| (1, -4) | ________ |
| (-4, 4) | ________ |
| (-3, -1) | ________ |
| (8, 6) | ________ |
| (7, -1) | ________ |
| (-3, -5) | ________ |

2. Plot the above points, using a rectangular system of coordinates. Label each point with its coordinates.

3. Plot the following points and connect them with line segments. What kind of figure is this?

   A(0, 5)  B(-3, -2)  C(3, -2)

4. Plot these points, using a rectangular system of coordinates:

   $(\frac{5}{2}, \frac{3}{2})$, $(-\frac{1}{2}, 7)$, $(0, \frac{7}{2})$, $(4, -\frac{3}{2})$, $(-\frac{5}{2}, -\frac{9}{2})$

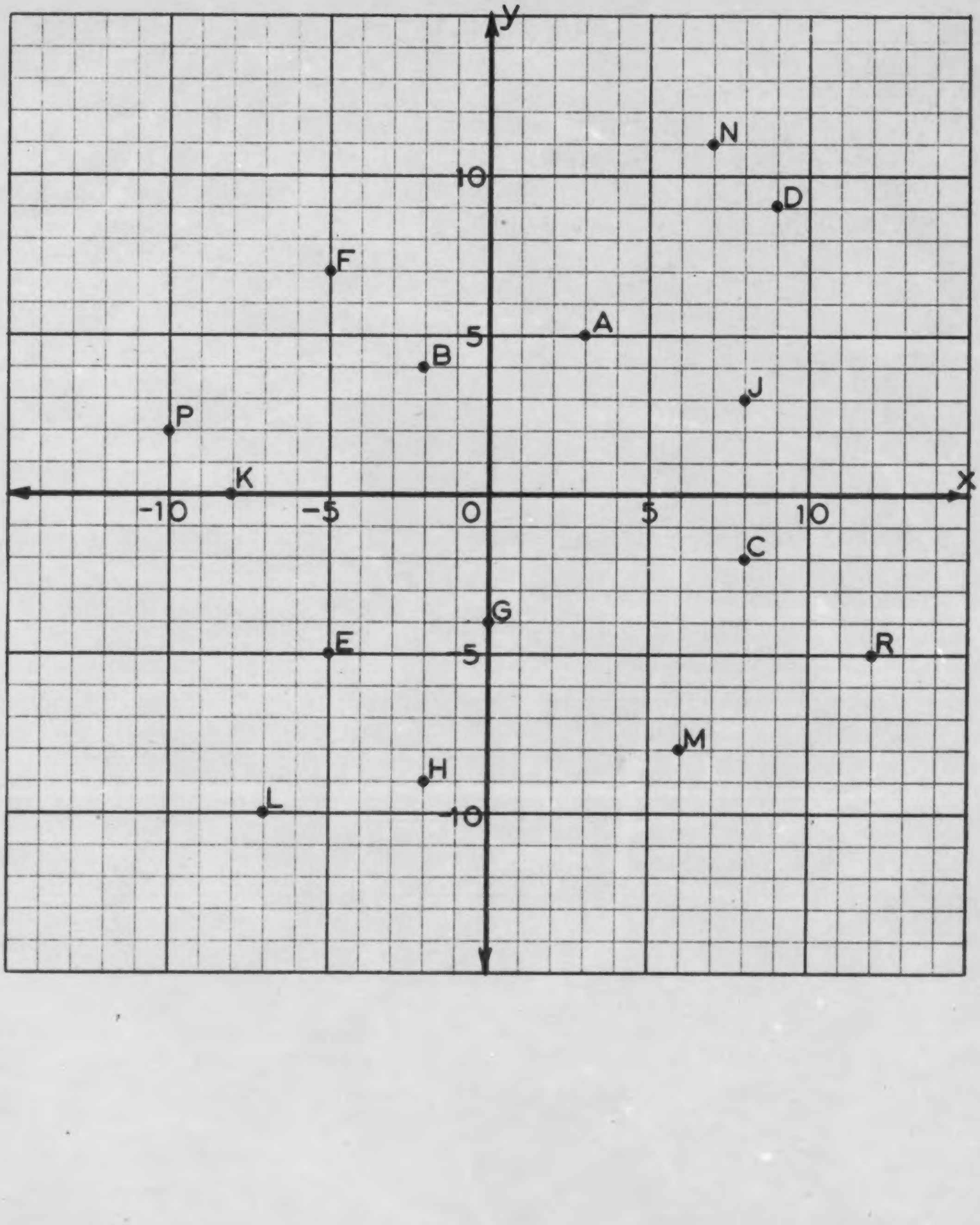
5. Write the ordered pairs associated with each of the following fifteen points. It is often desirable to number your scale by twos, threes, fives, etc., to save writing so many numbers on the scale. (See Figure on Page 37.)

6. (a) Both numbers of the ordered pair of coordinates are positive. The point is in Quadrant __________.

   (b) Both numbers of the ordered pair of coordinates are negative. The point is in Quadrant __________.

   (c) The x-coordinate is positive and the y-coordinate is negative. The point is in Quadrant __________.

   (d) The x-coordinate is negative and the y-coordinate is positive. The point is in Quadrant __________.


y
x
10
5
0
-5
-10
-10
-5
5
10
N
D
F
A
B
J
P
K
C
G
E
R
M
H
L

# UNIT 2

# EQUATIONS

## 2-1. Finding the Unknown

Do you like mystery stories? Have you every imagined yourself to be a detective like Sherlock Holmes or Nancy Drew? A mathematician often works like a detective, trying to solve a mystery, to find one or more unknown numbers from certain clues.

For example, I am trying to think of a certain number, Call it x (we often use letters like "x", "y", "r", to stand for unknown numbers). I give you the following clues

$$x + 5 = 7,$$

in words, 7 is 5 more than the unknown number. Can you detect what the number is? You probably can.

Sometimes there is more than one unknown. Here is a series of 4 numbers:

$$0, x, y, 300,$$

(two unknown and two known), and each of the "inside" numbers is the average of its two neighbors. What are the numbers? According to our clues

$$x = \frac{0 + y}{2} \quad \text{and} \quad y = \frac{x + 300}{2}$$

Can you tell what the unknown numbers are?

In both of these problems the clues were number sentences. Each clue was a statement about numbers, some known and some unknown. Since the verb in each of these sentences was the "equals" sign, we call such number sentences equations. We say that we are solving an equation, or a system of equations, for the unknowns x, y, etc.

Equations are used in many ways in many different fields. We solve equations to find the currents in an electrical network when we know the voltages and the resistances. We solve equations in order to design airplanes or space ships. We solve equations in order to find out what is happening in a cancer cell.

We also use equations to predict the weather. We now know methods for predicting tomorrow's weather very accurately. The only trouble is that these methods require the solution of about a thousand equations with the same number of unknowns. Even with the best of the modern high speed computers, it would take two weeks to compute the prediction of tomorrow's weather. Therefore the meteorologists (look up this word) make many approximations. They simplify the equations in such a way that they can compute the prediction in time. They will be able to make better predictions when we know more efficient ways to solve many equations with many unknowns.

Our progress in many fields of knowledge depends on finding better methods for solving equations. Many leading mathematicians are working on such problems. The National Bureau of Standards held two big conferences in 1953 and 1954 on new methods for solving equations.

If you had only to solve one simple equation such as $x + 7 = 9$, you might do it by trial and error. If you had many such equations to solve, you would try to discover a method for solving them. You might even try to design a machine for carrying out your method. When you can make a machine for solving all problems of a certain kind, then you can have the machine do the routine work for you. You then can reserve your brain for things which really require intelligence, such as solving new problems. When you finish this unit you should see that equation solving is not a lucky hit-or-miss activity which depends on trial and error.

### Exercise 2-1

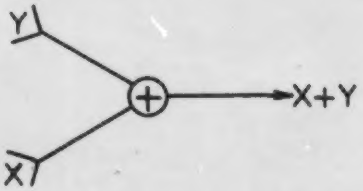
1. Bring in a book on one of the following subjects, which shows the use of equations. List at least two equations from the book.

   Physics, chemistry, biology, medicine, engineering, economics, or psychology.

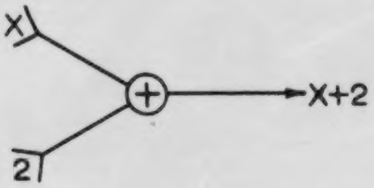
## 2-2. Designing Computers

Suppose that you are the designer in a company that makes computing machines. Your company has four basic machine parts or components, an adder, a subtracter, a multiplier, and a divider. Each of these components can be thought of as a simple computer which can perform one of the elementary operations. You have to tell your technicians how to put these together to make more complicated machines. In your diagram you show how these parts or components are connected up. You do not bother to make a more detailed diagram to show, for instance, how an adder is constructed.

You can use a symbol like this:

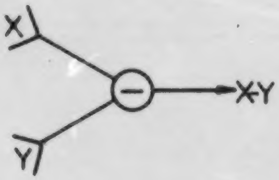


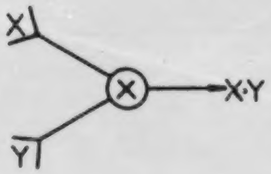
for an adder. There are two places where numbers are put in. These numbers are naturally called inputs. We have labeled the inputs with the letters x an y in our diagram. Inside the circle we have written the symbol for what the component does. When numbers are put in at x and y and the machine or component is set in operation, out comes the sum of the inputs. We call x + y the output of the machine. You may think of the lines in the diagram as representing wires along which signals are sent, or channels along which messages are sent. The numbers may be fed in by typing them on tape, or on punch cards. The outputs may be in similar form. This diagram shows a machine for adding x and 2.

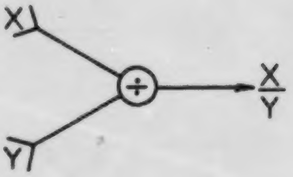


We have a constant input of 2 at one place. Whatever number x is put in at the other place, we get x + 2 as the output. If the input is 5, what is the output? If the output is -1, what must the input be?

Here are symbols for the other basic components:

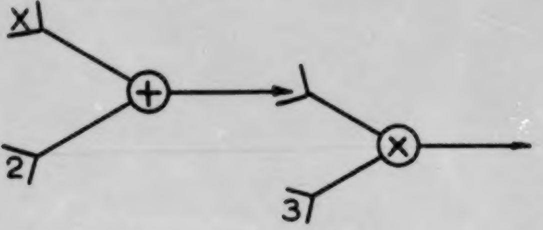






Notice in the subtracter and the divider which input is placed above the other.

Let us hook these components up in different ways and see what we get. Look at this diagram:



If the input is 4, what is the output?

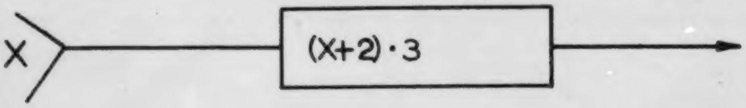
We could make a table:

| Input<br>x | Output |
|---|---|
| 0 | 6 |
| 1 | 9 |
| 2 | 12 |
| 5 | |
| 8 | |
| -1 | |
| -2 | |
| | 18 |
| | 45 |
| | -6 |

Fill in the empty spaces. For example, if x = 2, then the output of the adder is 2 + 2, which is _____. The output of the adder is one of the inputs of the multiplier and the other input is 3. Do you see why the output is 12?

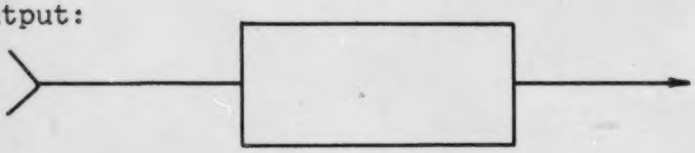
We can describe the operation of the above machine in a simple way. If the number x is the input, then the output of the adder is x + 2. When this number comes into the multiplier, together with the other input 3, the multiplier produces $(x + 2)\cdot 3$. We can summarize by saying that if the input is x, then the output is $(x + 2)\cdot 3$. Thus if x = 2, the output is $(2 + 2)\cdot 3 = 4\cdot 3 = 12$.

This machine might be part of a larger machine. We might not want to show all the details in designing the big machine. We might simply use a diagram like this

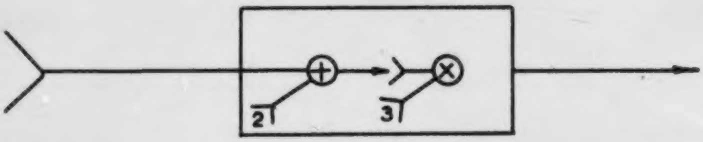


to show what this machine does.

Your company might manufacture the whole gadget as a unit. All the user sees is a box with a place for the input and a place for the output:



The buyer would not see the mechanism:

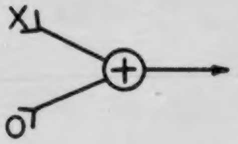


You might label the input with the letter x and the output with the expression $(x + 2)\cdot 3$, so that the user will know what comes out if he puts in any number.

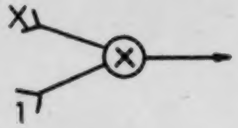
## Exercises 2-2a

1. Describe the outputs of the machines diagramed below:

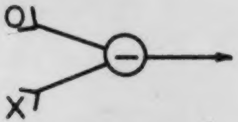
(a)



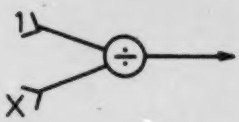
(b)



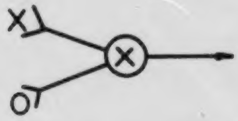
(c)



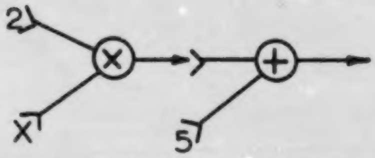
(d)



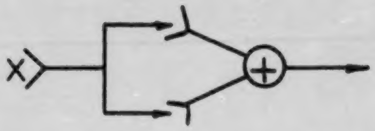
(e)



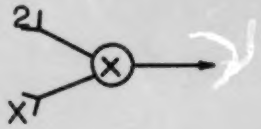
(f)



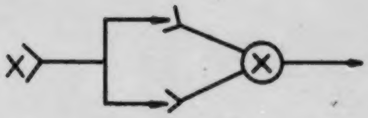
(g)



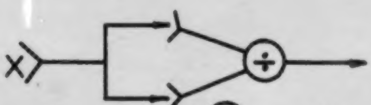
(h)



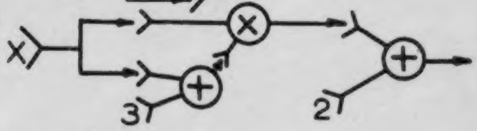
(i)



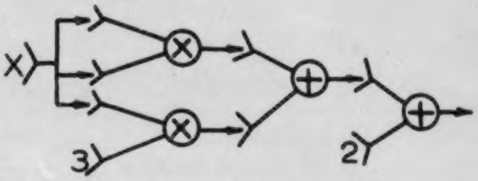
(j)



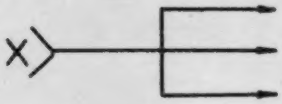
(k)



(l)



Notice that in machines (g), (i), (j), (k), and (l), and input x is transmitted along several channels. In this diagram if you put in 2, the number 2 comes out at all three



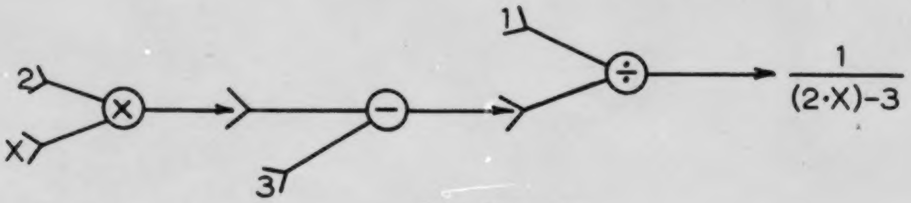
places where the arrows are.

2. For each of the above machines, tell what the output is for each of the following inputs: x = 1, -2, 3, and 0. Will all these machines accept the input 0?

3. For each of the machines (a) through (j), tell what number must be put in if you wish to obtain the output 9. Can all of them have the output 9?

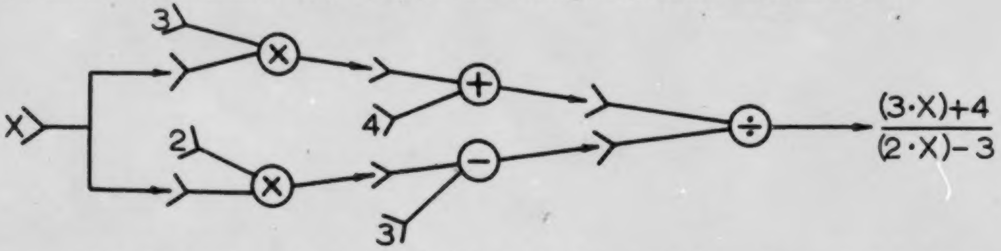
Suppose that you have to design a machine so that for the input x the output is $\frac{1}{(2\cdot x) - 3}$

If x is the input, the machine must first multiply x by 2, subtract 3 from the result, and then divide 1 by that difference. Here is a diagram for a machine to do this job:



If you want to compute $\frac{(3\cdot x) + 4}{(2\cdot x) - 3}$

you must transmit the input x along several channels:



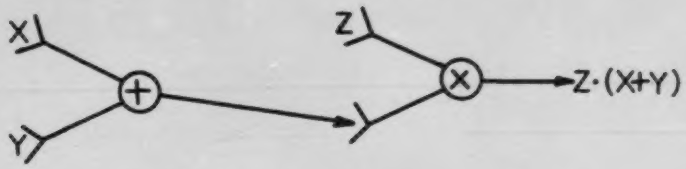
We could describe the work that this machine does like this:

$$x \longrightarrow \frac{(3\cdot x) + 4}{(2\cdot x) - 3}$$

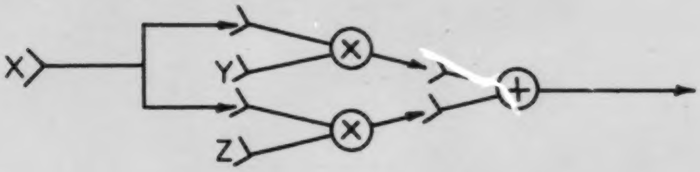
This shows that if you put the number x into the machine you get out the number on the right. For example, if you put in 7, you obtain

$$\frac{(3\cdot 7) + 4}{(2\cdot 7) - 3} = \frac{21 + 4}{14 - 3} = \frac{25}{11}$$

Our basic components are four machines each with two inputs and one output. We can design machines with even more inputs:

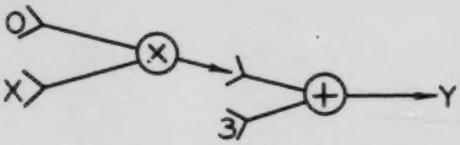


What is the output of this one?



Try putting numbers into each of these machines, and see what you get. Do you notice anything? What is the reason for what you notice?

Notice that sometimes there was a constant input, an input which does not change. The machine diagrammed below has a constant output. What is the value for y? Does it make any difference what number x you put into it?



Sometimes we have a variable input. We have been using letters like "x", "y", and "z", to indicate variables. In the machine for computing $(x + 2)\cdot 3$, the letter x can stand for any number. In each particular case it is the name for a certain definite number, but we may vary this number from time to time.

Sometimes a variable is restricted. If d is a length, then it cannot be a negative number. If n is an input to an ordinary adding machine, then n may only be a counting number from 1 to 999,999. A computer may accept only inputs from a certain set of possibilities. The set of all possible values for a variable is called the domain of the variable. If d is a length then the domain of the variable is the positive numbers. The domain of the input of an ordinary adding machine is the counting numbers from 1 to 999,999.

## Exercises 2-2b

1. Design machines for each of the following jobs:

   (a) $x \longrightarrow 2 - x$

   (b) $x \longrightarrow 1 + \frac{x}{2}$

   (c) $x \longrightarrow 4 \cdot (x + 1)$

   (d) $x \longrightarrow \frac{2}{x - 1}$

   (e) $x \longrightarrow x^2 - 1$

   (f) $x \longrightarrow (x - 1)\cdot(x + 1)$

   (g) $x \longrightarrow (x^2 + 1)\cdot x$

   (h) $x \longrightarrow \left(x^3 + (2\cdot x^2)\right) + 4$

In (g) and (h) you must be careful to pair the parentheses with each other properly.

2. For each of the above machines, tell what the output is for the following inputs: $x = 0, 1, -2$, and 10. Will all of these machines accept all of these inputs?

3. Design machines for computing from the inputs x and y the numbers:

   (a) $(x + y)\cdot(x - y)$

   (b) $x^2 - y^2$

   (c) $(x + y)^2$

   (d) $\left(x^2 + (2\cdot x\cdot y)\right) + y^2$

4. For each of the machines in exercise 3, tell what the output is when the inputs are the following pairs of numbers:

| x | 1 | 2 | 10 | 30 |
|---|---|---|---|---|
| y | 2 | 1 | 1 | 5 |

5. What is the domain of each of the following variables:

   (a) w = number of units of weight of a block of wood

   (b) v = speedometer reading of an automobile

   (c) M = the amount rung up on a cash register, in cents

   (d) T = number of units of temperature

   (e) p = number of people in a country

## 2-3. Number Sentences

Look at this sentence:

Jimmy was at Camp Jolly all day yesterday. Is it true or false? You may answer, "I don't know. Which Jimmy do you mean? I can't tell until I know who Jimmy is."

You may look at the records of the camp, and then reply, "It is true if you mean Jimmy Mills of Denver or Jimmy Good of Baltimore or Jimmy Schultz of Cincinnati. It is false for any other Jimmy."

Now think about this sentence:

If Jimmy was at Camp Jolly all day yesterday, then he was not at home at that time. Is it true or false? You may very well say, "It is true. I don't have to know who Jimmy is."

Now consider this sentence:

$$x + 3 = 8$$

Here "x" stands for some number, just as "Jimmy" stood for a boy in the previous discussion. Is this sentence true or false? You may answer "I don't know. I cannot tell until I know what number x is."

You may think a bit more and say, "If $x = 5$, then the sentence is true. If x is any other number, then it is false.

Look at this one now:

$$x + 3 = 3 + x$$

Is this true or false? You may answer, "It is true, no matter what number x is, by the commutative property of addition."

Sentences about numbers are called number sentences. In mathematical language, the most common verbs are " = ", " < ", and " > ". Here are three number sentences:

$$3 = 2 + 4$$

$$3 < 2 + 4$$

$$3 > 2 + 4$$

Each one is either true or false. Which are true, and which are false?

Sentences with the verb " = " are called equations and those with the verbs " < " and " > " are called inequalities.

Another type of number sentence is called a proportion. A proportion states that two ratios, or quotients, are equal. Tom asks a number of students how they will vote in a school election. In order that his poll should give him a good prediction, the ratio of boys to the size of his sample should be the same as the ratio of boys to the entire student population. If there are 700 boys in a total of 1,500 students in the school and he has 30 students in his sample, then he sets up the proportion

$$\frac{\text{number of boys in the sample}}{30} = \frac{700}{1500}$$

How many boys should be in his sample?

When the ratio of pairs of quantities are equal, we say that the corresponding quantities are proportional. The amount that Sally earns as a baby sitter is proportional to the number of hours that she works. If she earns 2 dollars in 3 hours, how much will she earn in 12 hours? We set up the proportion

$$\frac{\text{earnings in 12 hours}}{12} = \frac{2}{3}.$$

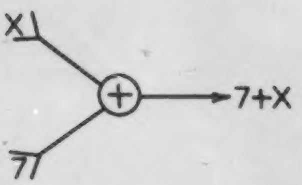
How much will she earn in 12 hours?

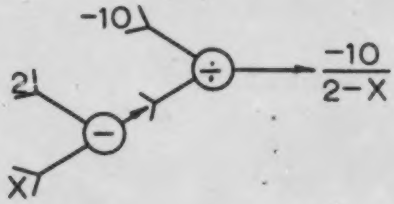
When you read a map, you often find a scale showing the ratio of the actual distances to the distances on the map. If 3 inches on the map correspond to 120 miles, then 7 inches corresponds to how many miles? Set up the proportion and solve the problem.

Expressions like $2 + 4$, $7 + x$, $(-10)/(2 - x)$ are number phrases. This first phrase is another name for 6. The true sentence $2 + 4 = 6$ expresses the fact that $2 + 4$ is the same number as 6. The second and third phrases are sometimes called open phrases, since we have not been told what x is. We have left the matter open for further consideration.

If $x = 5$, then $7 + x = 12$ and $-10/(2 - x) = 10/3$.

These open phrases describe the outputs of the machines





The phrase x + 7 describes the output of the first machine, whatever the number x may be.

Look at this sentence:

$$x + 7 = \frac{-10}{2 - x}$$

It states that the outputs are the same when a certain number x is the input. We have not said that this sentence is true. We have merely proposed it for discussion.

You cannot tell whether it is true until you know what x is. The matter has been left open. The sentence is called, then, an open sentence.

You may work on the sentence for a while and then say, "The sentence is true if x = 3 or x = -8. It is false if x is any other number. The numbers 3 and -8 are the solutions of the equation. If we solve the equation, we obtain x = 3 or x = -8."

You are not yet ready to solve equations as complicated as this one. You should be able to find the solutions of the equations in the following problems.

## Exercises 2-3a

1. Solve the following equations for the unknown number:

   (a) $x + 2 = 5$

   (b) $y + (-3) = 7$

   (c) $z + 3 = -7$

   (d) $t + (-3) = -7$

   (e) $2 \cdot u = 5$

   (f) $(-2) \cdot v = 7$

   (g) $3 \cdot w = -\frac{6}{5}$

   (h) $(-\frac{1}{2}) \cdot r = -\frac{2}{3}$

   (i) $7 - s = 2$

   (j) $7 - k = -2$

   (k) $\frac{14}{n} = -7$

   (l) $\frac{-30}{c} = 6$

   (m) $s - 7 = 2$

   (n) $k - 7 = -2$

   (o) $\frac{n}{14} = -7$

   (p) $\frac{c}{-30} = 6$

2. Write number sentences to describe the following situations:

   (a) Paul was 14 years old in 1958. In what year was he born?

   (b) I give you \$5. You now have \$13. How much did you have?

   (c) Vera is 3 times as tall as her baby brother Donald. She is 63 inches tall. How tall is he?

   (d) The Prairie Express travels at 80 miles per hour. How long does it take for a 500 mile trip?

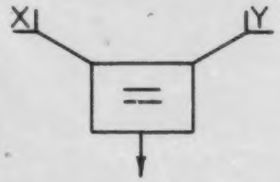
   (e) Your mother had $2\frac{1}{2}$ quarts of milk in the refrigerator before breakfast. Afterwards she found that she had $\frac{3}{4}$ of a quart left. How much did she serve?

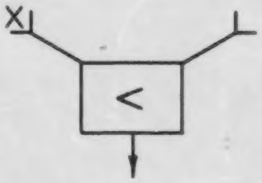
   *(f) The Grain Terminal had 100 toms of wheat. Some was sent to Buffalo at a price of \$90 a ton, and the rest was sent to Seattle at a price of \$100 a ton. The total receipts were \$9,400. How much was sent to each place?

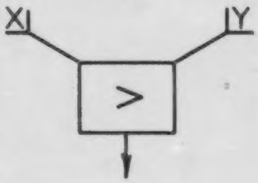
In these problems use letters to stand for the unknown numbers. You may not know how to solve all the equations.

3. BRAINBUSTER. The Wizard of Laputa wanted to know how many cows and how many chickens there were on a farm. First he counted the number of animals and found that there were 100. Then he counted the number of feet and found that there were 350 altogether. How many animals of each kind were there?

We may think of number sentences as related to truth machines. For each mathematical verb we can construct a component with two inputs and one output:





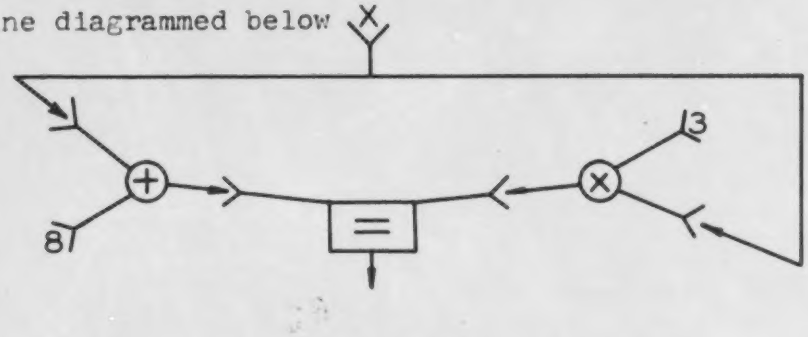


If numbers x and y are fed into any of these components the output is one of the symbols T or F (for "true" or "false"), for the truth value of the sentences

$$x = y, \qquad x < y, \qquad x > y$$

respectively. Thus if $x = 3$ and $y = 4$, the second component has the output T and the other two produce the output F.

The machine diagrammed below



will produce an output T or F whenever a number x is fed into the machine. The part on the left produces x + 8 and the part on the right computes $3 \cdot x$. The whole machine has the output T if the equation

$$x + 8 = 3 \cdot x$$

is true. Otherwise the output is F. Solving the equation means finding all numbers x for which the equation is true, that is, for which the output of the machine is T.

These numbers are called the solutions of the equation. The set of numbers is called the solution set of the number sentence (or equation)

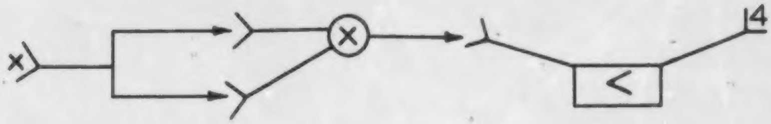
$$x + 8 = 3 \cdot x$$

**Find all the solutions of this equation.**

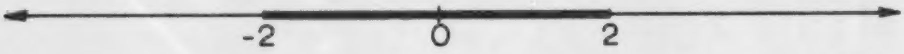
**The solution set of the number sentence**

$$x^2 < 4$$

**corresponding to the machine**



**is the set of all numbers x between -2 and 2.** It is represented by the segment on the number line.

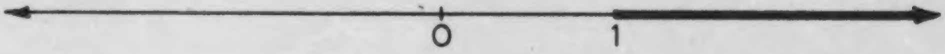


from -2 to 2, not including the end-points.

The solution set of the number sentence

$$(2 \cdot x) > x + 1$$

is represented by the half-line $x > 1$:



Remember that a half-line does not include the end-point.

## Exercises 2-3b

1. What are the solution set of the following number sentences? You may use the notation {3, 5, 11} as a name for the set whose only members are the numbers 3, 5, and 11.

   (a) $x + 2 = 7$

   (b) $2 \cdot x = 8$

   (c) $(2 \cdot x) + 1 = 7$

   (d) $\frac{x}{2} = 10$

   (e) $x^2 = 9$

   (f) $x^2 = 0$

   (g) $x^2 = -1$

   (h) $x^2 = 9$ and $x > 0$

   (i) $x + 1 = 1 + x$

   (j) $x + 1 = x + 2$

   (k) $x \cdot (x - 3) = 0$

   (l) $2 \cdot x < 10$

   (m) $x^2 > 16$ and $x < 100$ and $x > 0$

   (n) $x \cdot (1 - x) > 0$
   (Hint: When is the product of two numbers greater than zero?)

   (o) $x$ is an integer and $x^2 < 10$

   (p) $x$ is a positive integer and $x^2 < 10$

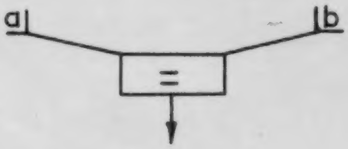
   (q) $x$ is a positive integer and $\frac{1}{x} < \frac{5}{8}$

   (r) $x$ is an integer and $3 \cdot x < 20$ and $4 \cdot x > 20$

## 2-4. Some Properties of Equations

Let us examine for a moment how one equation may be related to other equations. This will help us in solving equations. Some of the properties of equations are so obvious that you may wonder why anyone would bother to mention them. When you see what these properties lead to, you will realize that you can obtain surprising consequences from a very obvious beginning.

Suppose you feed $a = 3$ and $b = 3$ into the "equals" machine:



What is the output, T or F? If you replace 3 by any other number, will you obtain the same output? Surely, because any other number equals itself:

Property 1. If $a$ is any number, then $a = a$. (Reflexive property).

Suppose my job is to put in $a$ and yours is to put in $b$. Suppose we each put in a number, and the output is T. What will happen if we exchange numbers, if you feed in my number and I put in yours?

Property 2. If $a$ and $b$ are numbers and $a = b$, then $b = a$. (Symmetrical property).

If we read the equation

$$2 \cdot 3 = 6$$

from left to right, it expresses the result of a multiplication problem. If we read the equation

$$6 = 2 \cdot 3$$

from left to right, it expresses the solution of a problem in factoring.

Property 3: If a, b, and c are numbers, and $a = b$ and $b = c$, then $a = c$. (Transitive property).

Consider the rational numbers $\frac{20}{8}$, $\frac{10}{4}$, and $\frac{5}{2}$.

We first observe that

$\frac{20}{8} = \frac{10}{4}$ (dividing the numberator and denomiator of $\frac{20}{8}$ by 2)

and $\frac{10}{4} = \frac{5}{2}$ (dividing numerator and denominator of $\frac{10}{4}$ by 2)

but $\frac{20}{8} = \frac{5}{2}$ (dividing numerator and denominator of $\frac{20}{8}$ by 4).

Thus if $\frac{20}{8} = \frac{10}{4}$ and $\frac{10}{4} = \frac{5}{2}$, then $\frac{20}{8} = \frac{5}{2}$.

We usually use these properties of the equality relation without even thinking about them. Often we use these properties in our reasoning without mentioning them at all.

Because of the transitive property of equality we often use the abbreviation

$$a = b = c$$

for the number sentence $a = b$ and $b = c$.

Similarly, the following $a = b = c = d$ means $a = b$ and $b = c$ and $c = d$. Sometimes we conbine equations and inequalities:

$$a = b < c$$

in an abbreviation for $a = b$ and $b < c$. Such abbreviations cannot lead to confusion because of properties like the transitive property. We do not use the abbreviation

$$a > b < c$$

because if $a > b$ and $b < c$, there is no order relation between a and c which is always true.

Let us go back to the situation where you and I were operating the "equals" machine together. I am in charge of the "a" - input and you are in charge of the "b" - input. Each of us puts in a number, and we obtain the output T. Then each of us adds 5 to our original number, and feeds the sum into the machine. What will be the output now? We can express the general principle like this:

Property 4. If a, b, and c are numbers, and $a = b$, then $a + c = b + c$. (Addition property for equations.) and $c + a = c + b$.

Suppose that, instead of adding 5 to our original number, we had multiplied by 5. What would be the output then? We see that there is also a multiplication property for equations.

Property 5. If a, b, and c are numbers, and $a = b$, then $a \cdot c = b \cdot c$ and $c \cdot a = c \cdot b$ (Multiplication property for equations.)

Can you discover the subtraction and division properties for equations? Be careful! Remember that you cannot divide by zero.

We can apply these properties to the solution of equations. Suppose we wish to solve the equation

$$(2 \cdot x) + 3 = 17.$$

We can apply the addition property (using $z = -3$):

$$\text{If } (2 \cdot x) + 3 = 17, \text{ then } \big((2 \cdot x) + 3\big) + (-3) = 17 + (-3).$$

On the left, we use the associative property of addition and the fact that $3 + (-3) = 0$ and we obtain that if

$$(2 \cdot x) + 3 \quad + (-3) = 17 + (-3), \text{ then } 2 \cdot x = 14.$$

We can now use the meaning of division:

$$\text{if } 2 \cdot x = 14, \text{ then } x = \frac{14}{2}.$$

Of course, anyone knows that $\frac{14}{2}$ = _____ (What is it? You fill in the blank).

Thus we find that if

$$(2 \cdot x) + 3 = 17,$$

then

$$x = 7.$$

We have proved that if x is a solution of the given equation, then x must be 7. We still don't know that 7 is a solution. Let us see.

$$\begin{array}{lrl} \text{If} & x & = 7, \\ \text{then} & (2 \cdot x) + 3 & = (2 \cdot 7) + 3 \\ \text{but} & (2 \cdot 7) + 3 & = 14 + 3 \\ \text{and} & 14 + 3 & = 17 \\ \text{Therefore, if} & x & = 7, \\ \text{then} & (2 \cdot x) + 3 & = 17. \end{array}$$

Now we know that 7 is a solution of the equation. Since we proved first that there can't be any other, then 7 is the solution of the equation.

First we proved a uniqueness statement, showing that there are no more than a certain number (actually 1) of possibilities for solutions. Then we proved an *existence* statement, showing that a certain number *is* a solution.

Show where, in the above reasoning, we used properties 1 to 5. Notice that in several places we did several steps at once. When you are learning for the first time how to solve equations, it is a good idea to give the reason for each step. This helps you to avoid silly mistakes.

You may wonder how we thought of adding (-3) in the first step of solving the above equation. We thought, "Some number + 3 = 17. To find the number we must *undo* the operation of adding 3. We can accomplish this by adding -3." You may know another way to undo addition. Try to see whether you can solve the problem in another way.

## Class Exercise 2-4

1. Indicate which property, 4 or 5, is used in solving the following equations.

   (a) $v + 5 = 7$

   (b) $6.2 + y = 1.12$

   (c) $-2 + u = -10$

   (d) $5 \cdot x = 15$

   (e) $6 = \frac{x}{18}$

   (f) $14 - x = 0$

   (g) $\frac{1}{2} \cdot m = 17$

   (h) $18 + q = 8.6$

   (i) $x + 6 = 5 + 3$

   (j) $.08d = 73$

   (k) $19 = 6 - y$

   (l) $\frac{2}{3} \cdot n = 15 + .4$

   (m) $45 \cdot b = 1$

   (n) $\frac{7}{c} = 1$

2. What is done to the first equation to make the second?

Example: (1) $(2 \cdot x) + 4 = 7$

$(2 \cdot x) = 3$, Number 4 with $(-4)$

(a) (1) $2 \cdot (y + 4) = 8$

(2) $y + 4 = 4$

(b) (1) $1.6 = 4y$

(2) $.4 = y$

(c) (1) $\frac{2 \cdot (m + 5)}{3} = 6$

(2) $2 \cdot (m + 5) = 18$

(d) (1) $-x = 5$

(2) $x = -5$

(e) (1) $(\frac{1}{8} \cdot k) + 1 = 1$

(2) $\frac{1}{8} \cdot k = 0$

(f) (1) $\frac{2}{5} \cdot x = 10$

(2) $\frac{1}{5} \cdot x = 5$

(g) (1) $\frac{4}{n} = -26$

(2) $4 = (-26 \cdot n)$

(h) (1) $(.3 \cdot m) - 7.2 = 5$

(2) $(3 \cdot m) - 72 = 50$

(i) (1) $(5 \cdot x) - 2 = (3 \cdot x) + 6$

(2) $(2 \cdot x) - 2 = 6$

3. Use the properties as indicated on the input and output:

| | |
|---|---|
| Example: $5x - 7 = 2x$ | No. 4 with $(-2x)$ |
| (a) $y - 2 = 7$ | No. 4 with $(2)$ |
| (b) $3 - (2 \cdot y) = -5$ | No. 5 with $(-1)$ |
| (c) $(2 \cdot w) + 7 = (5 \cdot w) + 1$ | No. 4 with $(-2w)$ |
| (d) $7 = (3 \cdot w) + 1$ | No. 4 with $(-1)$ |
| (e) $6 = 3 \cdot w$ | No. 5 with $(\frac{1}{3})$ |
| (f) $\frac{t}{2} - 1.7 = -1.3$ | No. 5 with $(2)$ |
| (g) $b = \frac{x}{18}$ | No. 5 with $(18)$ |

(h) $(2 \cdot x) - (3 \cdot y) = 7$ No. 4 with $(3 \cdot y)$

(i) $y = 7 - (2 \cdot x)$ No. 4 with $(2 \cdot x)$

(j) $y = \frac{2}{x} - 3$ No. 5 with $(x)$

(k) $x \cdot y = 2$ No. 5 with $(\frac{1}{x})$

## Exercises 2-4

1. Solve the following equations by using the properties of the equality relation. Give your reason for each step.

   (a) $(2 \cdot x) + 1 = 7$

   (b) $y - 2 = 6$

   (c) $\frac{t}{2} - 3 = -4$

   (d) $(3 \cdot x) - 5 = (2 \cdot x) + 3$

2. Solve the following equations:

   (a) $x + 3 = 5$

   (b) $3 + y = -5$

   (c) $(2 \cdot v) + 3 = 5$

   (d) $3 + (2 \cdot m) = -5$

   (e) $(2 \cdot w) + 7 = (5 \cdot w) + 1$

   (f) $15 + (2 \cdot w) = (-5 \cdot w) + 1$

   (g) $y - 3 = 5$

   (h) $3 - u = -5$

   (i) $(2 \cdot w) - 3 = 5$

   (j) $3 - (2 \cdot s) = -5$

   (k) $(2 \cdot t) - 11 = (5 \cdot t) + 1$

   (l) $15 - (5 \cdot w) = (2 \cdot w) + 1$

3. (a) Which of the Properties 1 - 5 are also true of inequalities? Replace the " = " sign by the " < " sign in each, and tell whether it is still true or not. If it is not true, give examples with numbers in which it is false.

   (b) In Properties 1 to 3, replace the word "numbers" by "human beings" and the " = " sign by the phrase "is the father of". Which of the resulting statements are true?

   (c) Do the same as in part (b) using the phrase "is the ancestor of" in the place of the " = " sign.

(d) Do the same as in part (b), using the phrase "is the same person as or is an ancestor of".

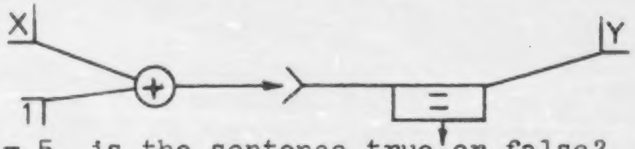
(e) Do the same as in part (b), using the phrase "is married to".

## 2-5. Number Sentences with Two Inputs. Graphs.

In the previous examples of number sentences, there was only one input. We could also have more than one input. Look at this sentence

$$x + 1 = y.$$

A truth machine for this sentence would look like this



If $x = 3$ and $y = 5$, is the sentence true or false? If $x = 7$, what must $y$ be for the sentence to be true? If $y = -6$ what must $x$ be in order that the sentence be true?

The solutions of this equation are pairs of numbers. We can make a table listing some of these pairs:

| x | y |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | |
| | 0 |
| $-\frac{2}{3}$ | |
| | $-\frac{13}{3}$ |

Before you continue reading, copy this table and work out the missing numbers. For example, to fill in the third line, set $x = 2$ in the above equation. Ask yourself, "What are the possible values of $y$?" You must read the rest of this chapter with pencil and paper handy. Do not go on to a new paragraph until you have answered all the questions in the paragraph you have just read. In much of this chapter you will need to use graph paper and a ruler, too.

We may say that the pair (0, 1) is a solution of the equation. This means that if $x = 0$ and $y = 1$, then the equation is true. Notice that it makes a difference which number is named first. The pair (1, 0) is not a solution since if $x = 1$ and $y = 0$,

then $$x + 1 = 1 + 1 = 2$$

and the equation $$x + 1 = y$$

is not true.

A pair in which the objects are considered in a definite order is called an ordered pair.

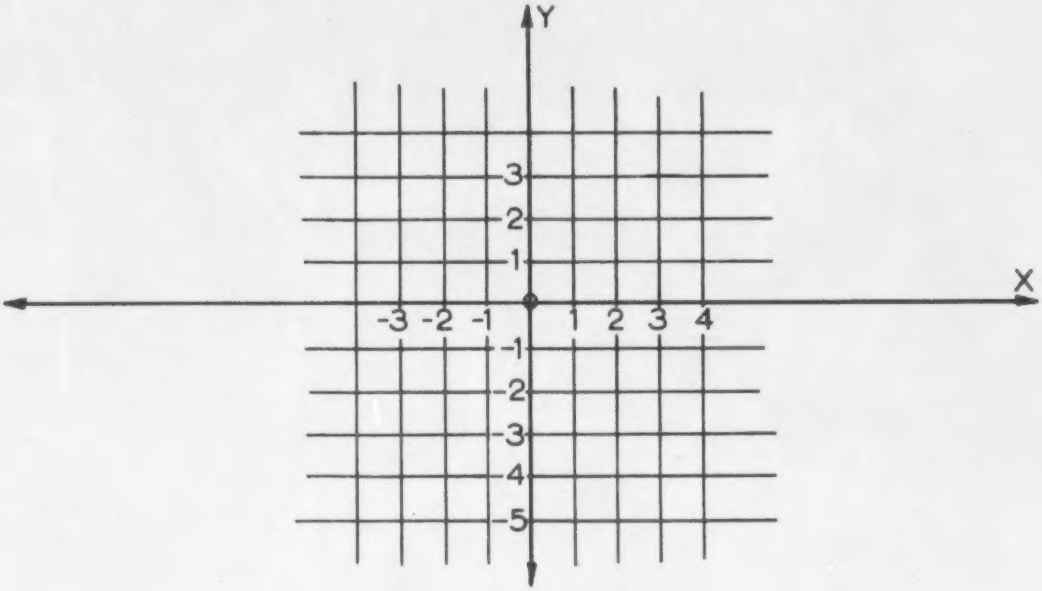
The ordered pair (2, 7) is the same as the pair $(x, y)$ if $x = 2$ and $y = 7$, and only then. This pair is different from the ordered pair (7, 2).

The solution set of the above sentence

$$x + 1 = y$$

is a set of ordered pairs of numbers. For which number $y$ is the ordered pair $(2, y)$ in the solution set?

Picture the solution set on your graph paper. Pick out two lines for the x-axis and the y-axis and draw them in heavily with your pencil. Label the vertical and horizontal lines as shown.



Mark off on your graph paper all the points (0, 1), (1, 2), etc., whose coordinates are in the solution set. What do you notice about them? They form a simple geometric figure. To what set of points does the solution set correspond? This set of points is called the graph of the given number sentence, or equation.

We are going to illustrate with the equation

$$y = x^3.$$

We first make a table of solution:

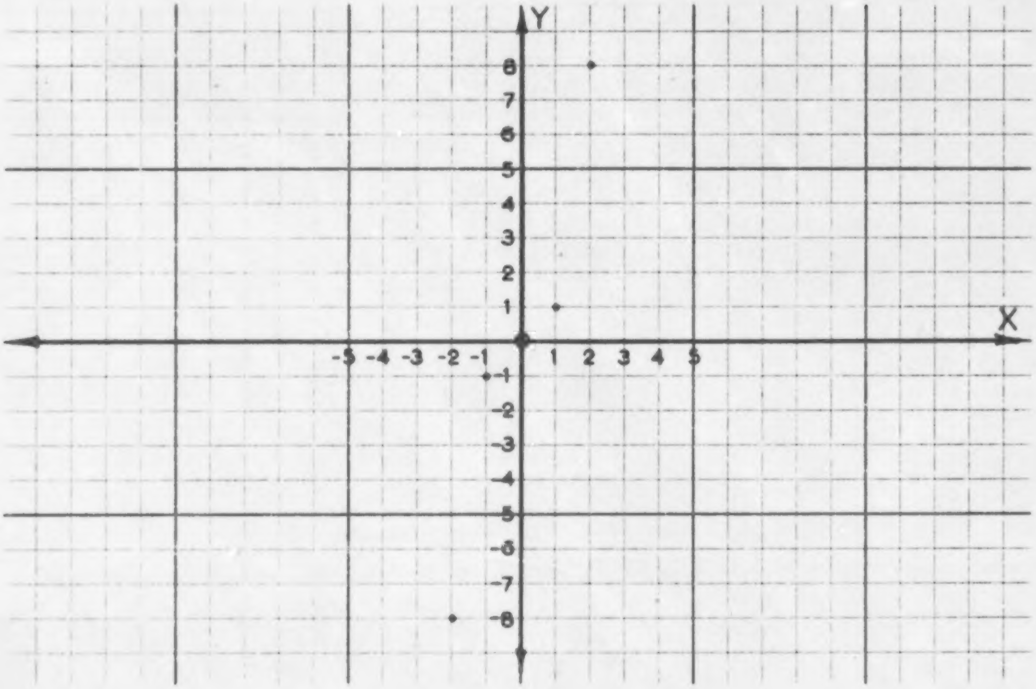
| x | y |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |

We can see that as x increases, y increases rapidly. What happens if x is negative?

| x | y |
|---|---|
| -1 | 1 |
| -2 | -8 |
| -3 | -27 |

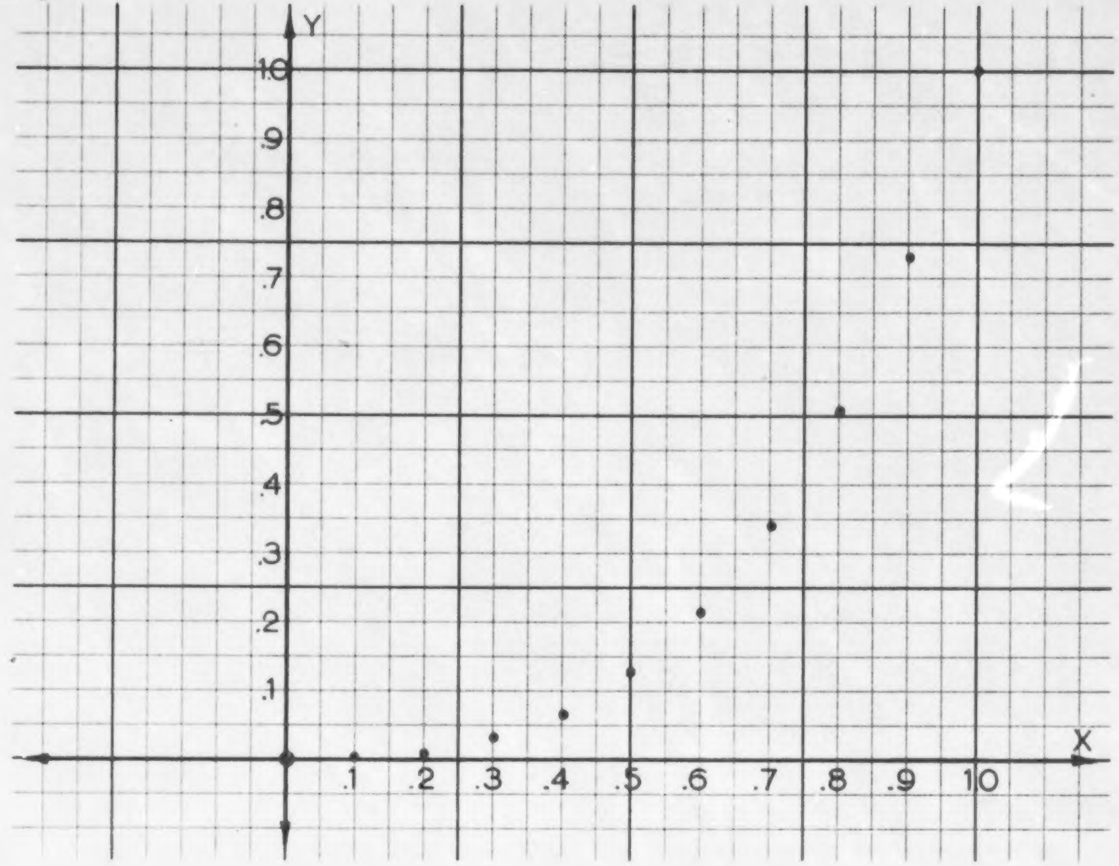
Do you notice anything?

Let us plot these points on our graph paper:

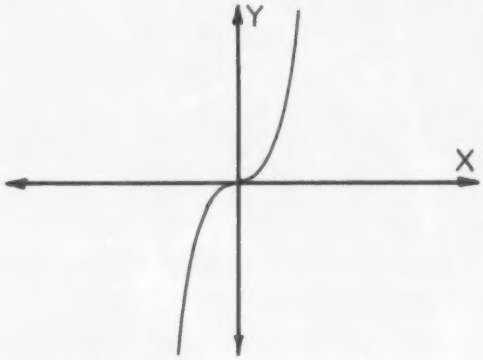


Already the points (3, 27) and (-3,-27) are off our graph paper. But we can see that the graph is rising rapidly on the right. How do you think it goes on the left? The points (-1, -1) (0, 0) and (1, 1) are mysterious. Is this part of the graph straight or curved? In order to see more clearly what the graph looks like, let us use a magnifying glass, let us draw this part of the graph with a much bigger scale and find some more pairs in the solution set.

| x | y |  |
|---|---|---|
| 1 | 1 |  |
| .9 | .729 |  |
| .8 | .512 |  |
| .7 | .343 |  |
| .6 | .216 |  |
| .5 | .125 | (CHECK these computations) |
| .4 | .064 |  |
| .3 | .027 |  |
| .2 | .008 |  |
| .1 | .001 |  |



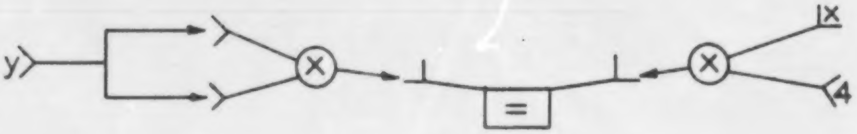
We see that the graph is a continuous curve that looks like this:



Just for practice, let us draw the graph of the equation

$$y^2 = 4x$$

Here is a diagram of the truth machine for this number sentence:



Fill in the blanks in the following table:

| x | y |
|---|---|
| 0 | 0 |
| 4 | 4 |
|  | -4 |
|  | 1.9 |
|  | 2.1 |
| 9 |  |
| -1 |  |

Find five more solutions of the equation. Plot the points and sketch the graph. Where does this graph intersect the graph of the equation $x + 1 = y$? Where does it intersect the graph of the equation $y = x$?

Let us try another one, just for fun. Bonnie has in her purse 3 dollars in dimes and quarters. What possible combinations can she have?

Let $d$ be the number of dimes and $q$ be the number of quarters. Just as the value of 3 dimes is $10 \cdot 3$ cents, so the value of $d$ dimes is $10 \cdot d$ cents. Similarly, the value of $q$ quarters is $25 \cdot q$ cents. The total value of these coins is $(10 \cdot d) + (25 \cdot q)$ cents, and this must be equal to 3 dollars.

But, wait a minute! We must make up our mind whether we want to measure our money in cents or dollars. Let us use cents throughout. Then 3 dollars is 300 cents. Therefore the pair (d,q) is a solution of the equation

$$(10 \cdot d) + (25 \cdot q) = 300.$$

We must be careful, however. This equation is not a completely correct translation of the real situation into mathematical language. Bonnie cannot have twenty-seven and one-half dimes. The inputs of this problem must be non-negative integers. The number sentence which really describes the situation is, $d$ and $q$ are non-negative integers and $(10 \cdot d) + (25 \cdot q) = 300$. The solution set of this number sentence is made up of the following 7 ordered pairs:

$$(0, 12), (5, 10), (10, 8), (15, 6), (20, 4), (25, 2), (30, 0).$$

The graph of this number sentence consists of seven isolated points.

A chain store has 5 tons of coffee in its warehouse in New Orleans. It sends $s$ tons to San Francisco and $n$ tons to New York. What are all the possibilities?

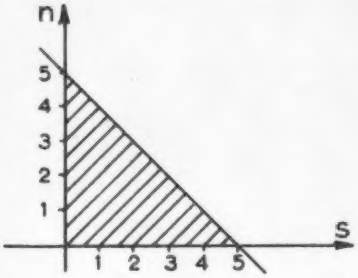
The total amount sent is $s + n$ tons. This cannot be more than there is in the warehouse altogether. Therefore $s$ and $n$ are related by the inequality

$$s + n \leq 5.$$

(Remember that the symbol " $\leq$ " means "is equal to or less than.") But (-1, 6) is a solution of this number sentence which does not fit the problem. How can you send -1 tons of coffee anywhere? The inputs $s$ and $n$ must be non-negative. The correct mathematical description of the relation between s and n is the number sentence

$$s \geq 0 \text{ and } n \geq 0 \text{ and } s + n \leq 5.$$

Let us try $s = 2$ and find all the possible values for $n$. Since $2 + n \leq 5$, then $n$ cannot be more than 3. But $n$ cannot be negative either. Therefore, $0 \leq n \leq 3$ and it is easy to see that any number between 0 and 3, the extreme is included, (0 and 3 is included) is a possible value for n. The points $(2, n)$, with $0 \leq n \leq 3$, form the segment from (2, 0) to (2, 3).



Similarly we see that if $x = 3$, the solutions are represented by the segment from (3, 0) to (3, 2).

For any number s between 0 and 5 the greatest possible value of n is $5 - s$, and the least possible value is $n = 0$ The number n can have any value between these extremes. The number s cannot be more than 5 and it cannot be negative. So the boundary of the graph of the above number sentence is part of the lines $n = 5 - s$, $n = 0$, and $s = 0$. The graph of the number sentence is the triangular region enclosed by these lines, together with the boundary of the region.

## Exercises 2-5

1. Graph the following equations:*

(a) $y = x + 1$
(b) $y = (2 \cdot x) + 1$
(c) $y = (3 \cdot x) + 1$
(d) $y = (-2 \cdot x) + 1$
(e) $y = x + 2$
(f) $y = x + (-3)$
(g) $x + y = 0$
(h) $x + y = 1$
(i) $x + y = -1$
(j) $\frac{x}{2} + \frac{y}{3} = 1$
(k) $\frac{x}{3} - \frac{y}{2} = 1$
(l) $y = (2 \cdot x) + 3$
(m) $y = (-\frac{1}{2} \cdot x) + 3$

*Draw the following families of graphs on the same sheet of graph paper:

(a), (b), (c), (d)
(a), (e), (f)
(g), (h), (i)
(j), (k)
(a), (h)
(l), (m)

2. Graph the following equations:*

(a) $y = x^2$
(b) $y = x^5$
(c) $y^2 = x$
(d) $y = - x^2$
(e) $y^2 = x^2$
(f) $x \cdot y = 1$
(g) $x \cdot y = -1$
(h) $x \cdot y = 0$

*Draw the following families of graphs on the same sheet of graph paper.

(a), (b); (a),(c); (a),(d); (e),(f),(g),(h)

3. Graph the following number sentences:

(a) $x + y = 1$ and $x \geq 0$ and $y \geq 0$
(b) $x + y \leq 1$ and $x \geq 0$ and $y \geq 0$
(c) $x + y = Lo$ and $x$ and $y$ are non-negative integers.

(d) $y = x$ when $x \geq 0$ and $y = -x$ when $x < 0$.

(e) $y = x^2$ and $y \leq 1$.

(f) $y$ = the larger of the numbers $x + 1$ and $2 - x$.

(g) $y = 1$(Hint: this is the same as $y = 1 + (0 \cdot x)$

(h) $y^2 = 1$.

(i) $x = 1$.

(j) $x^2 = 0$

(k) $x = 0$ and $y = 0$.

(l) $x^2 + y^2 = 0$.

4. In the following number sentences the domain of the variable is the set of non-negative integers. List the solution sets. How many solutions does each of the sentences have?

(a) $x + y = 1$

(b) $x + y = 2$

(c) $x + y = 20$

(d) $x + 2y = 0$

(e) $x + 2y = 1$

(f) $x + 2y = 2$

(g) $x + 2y = 3$

(h) $x + 2y = 4$

(i) $x + 2y = 25$

(j) $(5 \cdot x) + (7 \cdot y) = 35$

(k) $(5 \cdot x) + (7 \cdot y) = 36$

(l) $(5 \cdot x) + (7 \cdot y) = 37$

5. In the problem about the coffee, suppose that the warehouse must send at least one ton to San Francisco and 3 tons to New York in order to supply regular customers. Write a number sentence which describes the new situation completely. Graph this sentence.

6. BRAINBUSTER. Make a table showing all the solutions, and the number of solutions, of the number sentence.

x and y are non-negative integers and $(2 \cdot x) + (3 \cdot y) = n$ for n = 0, 1, 2, 3, etc. Here is the beginning of the table.

| n | Solutions | Number of Solutions |
|---|---|---|
| 0 | (0, 0) | 1 |
| 1 | | 0 |
| 2 | (1, 0) | 1 |
| 3 | (0, 1) | 1 |
| 4 | (2, 0) | 1 |
| 5 | (1, 1) | 1 |
| 6 | (3, 0), (0, 2) | 2 |
| 7 | (2, 1) | 1 |
| 8 | (4, 0), (1, 2) | 2 |
| 9 | (3, 1), (0, 3) | 2 |
| 10 | (5, 0), (2, 2) | 2 |

(a) Subtract from each number in the third column the one three lines above it. For example subtract from the number in the 3rd line the one in the 0th line, from the one in the 10th line the one in the 7th line, etc. What did you notice about these differences? Do you see any easy way to continue the table?

(b) For all the solutions in the n-th line compute x + y. Call this number k. For instance, in the 10th line we obtain values of k the numbers 5 + 0 = 5 and 2 + 2 = 4. What do you notice about the values of k in each line?

(c) Make a table showing the largest and smallest values of k in each line. Here is part of the table:

| n | largest value of k | smallest value of k |
|---|---|---|
| 6 | 3 | 2 |
| 7 | 3 | 3 |
| 8 | 4 | 3 |
| 9 | 4 | 3 |
| 10 | 5 | 4 |

What is the law about the largest value of k? What about the smallest value of k? How are these values related to the number of solutions? Can you find the largest and smallest values of k when $n = 100$? What is the number of solutions when $n = 100$?

(d) Investigate the same problem for some number sentence, such as; x and y are non-negative integers and $(3 \cdot x) + (4 \cdot y) = n$.

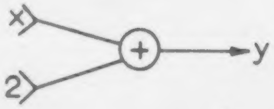
## 2-6. Solving Families of Equations

Solve the equations:

$$x + 2 = 3,$$
$$x + 2 = 4,$$
$$x + 2 = 5,$$
$$x + 2 = 6,$$

By now you must be getting bored. You may decide to invent a machine for solving all problems of this type.

You could imagine that there is already a machine for computing x + 2:



Someone feeds you the output y of this machine. You want to know what the input x was. You wish to design a machine that works on y and produces as its output the original input x.

If y = x + 2, what do you have to do with y to obtain x? Look at the above cases where y = 3, 4, 5, and 6, respectively. What must you do to undo adding 2? Can you make a machine which would operate on y and produce x?

Solve the following equations for x:

$$x + 7 = y$$
$$x - 2 = y$$
$$3 + x = y$$
$$5 - x = y$$

Check your solutions by choosing particular values for y, such as y = 4 or y = -2, and computing x.

Now solve the equations:

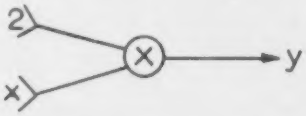
$$2 \cdot x = 0,$$
$$2 \cdot x = 1,$$
$$2 \cdot x = 2,$$
$$2 \cdot x = 3,$$
$$2 \cdot x = 4.$$

What is the general principle? If $y$ is known, how can you solve the equation

$$2 \cdot x = y$$

for x?

Design a machine for computing the input $x$ from the output $y$ of the machine diagrammed below



Solve the following equations for $x$:

$$3 \cdot x = y,$$

$$(-4) \cdot x = y,$$

$$\frac{x}{3} = y,$$

$$\frac{3}{x} = y,$$

$$x \cdot 3 = y$$

Now try solving these equations:

$$(2 \cdot x) + 3 = 0,$$

$$(2 \cdot x) + 3 = 4,$$

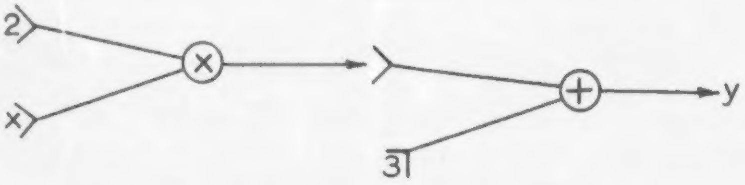
$$(2 \cdot x) + 3 = 5,$$

$$(2 \cdot x) + 3 = 6.$$

Can you solve this one for x:

$$(2 \cdot x) + 3 = y.$$

Design a machine for computing the input $x$ if you are given the output $y$ of this machine:



## Exercises 2-6

1. Solve the following equations for x:

(a) $y = (2 \cdot x) + 1$

(b) $y = (2 \cdot x) - 3$

(c) $(2 \cdot x) + y = 7$

(d) $(2 \cdot x) + (3 \cdot y) = 7$

(e) $y = \frac{x}{2}$

(f) $y = \frac{2}{x}$

(g) $x \cdot y = 2$

(h) $y = (2 \cdot x) + 5$

(i) $y = 7 - (2 \cdot x)$

(j) $(2 \cdot x) = 7 - y$

(k) $(2 \cdot x) - (3 \cdot y) = 7$

(l) $y = \frac{x - 1}{2}$

(m) $y = \frac{2}{x} - 3$

(n) $x \cdot (y + 3) = 2$

2. For each part in Problem 1, design a machine for computing $x$ if the number $y$ is given.

3. (a) If the input to the machine in Figure 2-6 is $x = -3$, what is the output? Solve the equation

$$(2 \cdot x) + 3 = y.$$

(b) Solve the number sentence:

$y = (2 \cdot x) + 3$ and $y = x$.

(c) Solve the number sentence:

$y = (2 \cdot x) - 3$ and $y = x$

(d) Solve the number sentence:

$y = (2 \cdot x) - 3$ and $y = x + 1$.

(e) Graph the equations

$y = (2 \cdot x) - 3$

and

$y = x + 1$.

Locate the solution of part (d) on your graph. What do you notice?

4. BRAINBUSTER. Solve the equation: $x = \frac{1}{2} \cdot (x + \frac{4}{x})$.

## UNIT 3

# SCIENTIFIC NOTATION, APPLICATIONS OF PER CENT

### 3-1. Large Numbers

Harry, Edward and Edward's younger brother, Tim, were playing a game, showing off how much they knew about numbers. Harry started it by boasting "I know more numbers than you do." Of course, Edward responded with "I'll bet you don't", and little Tim, wanting to continue to enjoy their company, said nothing at all. So Harry started with "a hundred" and Edward came back with "a thousand is more than a hundred". Harry gained the advantage with "one million" and Edward, after some thought, remembered "one billion" (he had heard it in connection with the national debt). That was as far as they could go and Edward was just about to declare himself the victor when little Tim spoke up with "one more than one billion". For this the older boys pounced on him and sent him home. But after Tim had gone, they spent the rest of a warm summer afternoon (the fish were not biting) arguing about it. Do you think Tim was right? If you had been there would you have won the game? How?

Actually we have names for larger groups of numbers than one billion, such as trillion and quadrillion. Consider the numeral

$$3141592653589793.$$

It is hard to read such a number written in this form. One common way to make it easier to read is to place a comma to the left of every third digit counting from the right as follows:

3,141,592,653,589,793.

Though we put the commas in from right to left, we read the number from left to right according to the following diagram:

| | trillion | | | billion | | | million | | | thousand | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| one quadrillion | hundred | ten | one | hundred | ten | one | hundred | ten | one | hundred | ten | one | hundreds | tens | ones |
| 3 , | 1 | 4 | 1 , | 5 | 9 | 2 , | 6 | 5 | 3 , | 5 | 8 | 9 , | 7 | 9 | 3 |

Thus we read this number as follows: three quadrillion, one hundred forty-one trillion, five hundred ninety-two billion, six hundred fifty-three million, five hundred eighty-nine thousand, seven hundred ninety-three. In reading such a number we have to be careful not to use the word "and". We can see the reason for this if we consider what might be meant by "five hundred and ninety-three thousand". We associate the word "and" with addition and would write this in numerals: 500 plus 93,000, which is equal to 93,500. But five hundred nine-three thousand would 593,000, which is a much larger number. Omitting the "and" avoids misunderstanding. We usually use the "and" to mark the decimal point, e.g., 563.12 is read "five hundred sixty-three and twelve hundredths".

You may not know that the British have a different way of denoting large numbers. Their words "thousand" and "million" mean the same as ours but their "billion" means what we would call "a million millions" or "one trillion". They would call the above numeral the following:

Three thousand one hundred forty-one billion,
five hundred ninety-two thousand six hundred fifty-three million,
five hundred eighty-nine thousand,
seven hundred ninety-three.

Do you see any advantages in the British system? Any disadvantages?

Actually such numbers as these seldom occur. This does not mean that numbers of this size do not happen but merely that we do not know any number which is as accurate as this would seem to indicate. The population of a city of over a million inhabitants might be given as 1,576,961 but this just happened to be the sum of the various numbers compiled by the census takers. However, it is certain that the number changed while the census was being taken and that 1,577,000 would be accurate to within one person in a thousand--just as accurate proportionately as if a man had made an error of 1 in counting the population of a town of 1000 persons. For this reason there is no harm in rounding the original number to 1,577,000. In fact, for most purposes, we would merely say that the population of the city is "about one and one-half million, which could be written:

1,500,000.

There are other ways of writing this number which have some advantages over this one. A hint of how this can be done is given by our statement "one and one-half million". Now one million can be written: 1,000,000. Also it is

$$10 \times 10 \times 10 \times 10 \times 10 \times 10,$$

that is, the product of six tens. So we can write

$$1,000,000 = 10^6.$$

The "exponent" 6 counts the number of tens in the product. We could also get it by counting the number of zeros in the numeral 1,000,000. (Similarly one billion could be written $10^9$. Why? Just as one million is the sixth power of ten, so one billion is the ninth power of ten. How could one trillion be written as a power of ten?)

Then two million is two times one million and could be written

$$2 \times 10^6$$

Eleven million is $11 \times 10^6$,

One and one-half million is $1\frac{1}{2} \times 10^6$ or $1.5 \times 10^6$.

This form of writing numbers is called "scientific notation".

To take another example, consider the distance from the earth to the sun. You know that it varies according to the time of the year since the earth does not travel in a circular path; but the average distance has been calculated to be about 93,004,000 miles. It can vary by about 3%. That is, at any time the distance could be $1\frac{1}{2}$% more or $1\frac{1}{2}$% less. Now $1\frac{1}{2}$% of 93,004,00 is about one million (see Problem 4 below); that is, the distance can vary from 93,004,000 by more than a million miles in either direction.

Hence there is no great loss in accuracy if we say that the distance from the earth to the sun is about 93 million miles. We could write this $93 \times 10^6$. Since $93 \times 10^6 = 9.3 \times 10 \times 10^6$ this number could also be written:

$$9.3 \times 10^7.$$

This is another example of "scientific notation".

Definition. A number is expressed in scientific notation if it is written as a product of a number between 1 and 10 and the appropriate power of 10.

Some examples are:

$$900{,}000 = 9 \times 10^6 \qquad 70 = 7 \times 10^1 = 7 \times 10$$

$$8{,}000{,}000{,}000{,}000 = 8 \times 10^{12} \qquad 7800 = 7.8 \times 10^3$$

$$193{,}000 = 1.93 \times 10^5 \qquad 3457 = 3.457 \times 10^3$$

Of course there would be no harm in writing 193,000 as

$$1.930 \times 10^5 \text{ or } 1.9300 \times 10^5.$$

Although there are situations in which this would be done, we shall not do it in this unit. Notice that each is in scientific notation, since the members of the product are a power of ten and a number between 1 and 10.

## Exercises 3-1

1. Write the following in scientific notation:

(a) 678,000.

(b) 9,000,000,000.

(c) 5,000.

(d) 78,000

(e) 459,000,000.

(f) $781 \times 10^7$

2. Write each of the following in a form which does not use scientific notation:

   (a) $2.3 \times 10^4$ (b) $5.897 \times 10^8$ (c) $7,324 \times 10^5$

3. Write, or say, in words each of the numbers of the last two exercises.
4. What is $1\frac{1}{2}\%$ of 93,004,00?
5. Using just two 9's can you write a number larger than 99?
6. Write the largest number you can using just three 9's. Estimate how large it is.

## 3-2. Calculating with Large Numbers

Not only is scientific notation shorter in many cases but it makes certain calculations easier. We shall start with some rather simple ones. Suppose we want to find the value of the product: 100,000 × 1, 000,000. Since the first member is the product of five tens and the second of six tens we have

$$100,000 = 10^5 \quad \text{and} \quad 1,000,000 = 10^6.$$

Then, since the product of five tens and the product of six tens is eleven tens, we have

$$10^5 \times 10^6 = 10^{11}.$$

This is one hundred billion but it is simpler to leave it in the form $10^{11}$ than to write a 1 followed by eleven zeros. Notice that we merely add the exponents. Similarly,

$$2^5 \times 2^6 = 2^{11}.$$

Suppose we wish to find the product of 93,000,000 and 11,000. In scientific notation this would be

$$9.3 \times 10^7 \times 1.1 \times 10^4$$
$$= 9.3 \times 1.1. \times 10^7 \times 10^4$$
$$= 10.23 \times 10^{11}$$
$$= 1.023 \times 10^{12}.$$

Can you find another way to use powers of ten in such a product without quite using scientific notation? Do you think it is simpler than the one we used?

Now consider a more difficult example. Distances to the stars are usually measured in "light years", where a light year is the distance light travels in a year. This is a good way to measure such distances because if we expressed them in miles the numbers would be so large that it would be difficult to write them, much less understand what they mean. But suppose we wish to estimate about how many miles a light year is. Now, the speed of light has been determined to be about 186,284 miles per second. (Since the distance around the earth at the equator is about 25,000 miles, and since electricity has the same speed as light, can you calculate how many times around the earth electricity could travel in a second? See Problem 4 below.) Thus, in order to find the number of miles in a light year, we must first find the number of seconds in a year. Then if we multiply this number by 186,284, we will have the number of miles in a light year.

The number of seconds in a year is approximately:

$$60 \times 60 \times 24 \times 365 = 3600 \times 8760.$$

In scientific notation this is

$$\begin{aligned} & 3.6 \times 10^3 \times 8.76 \times 10^3 \\ = & \ 3.6 \times 8.76 \times 10^3 \times 10^3 \\ = & \ 31.536 \times 10^6 \\ = & \ 31{,}536{,}000. \end{aligned}$$

This is approximately 32 million, which is

$$3.2 \times 10^7$$

in scientific notation. (Why is 31,536,000 not exactly the number of seconds in a year?).

Now if we take the speed of light to be approximately 200,000 miles per second, then the number of miles in a light year will be approximately the product:

$$2 \times 10^5 \times 3.2 \times 10^7 = 6.4 \times 10^{12}.$$

Thus there are almost six and a half quadrillion miles in a light year. No wonder distances to the stars are expressed in light years instead of miles! But we shall see that this number is very small compared to the number of atoms it takes to weigh an ounce.

## Exercises 3-2

1. Using scientific notation, find each of the following products and express your answer in scientific notation:
   - (a) 9,000,000,000 × 70,000
   - (b) 9,300,000 × 72,000
   - (c) 125 × 17,800,000,000

2. Using the estimate we made above for the number of seconds in a year and the fact that sound travels about one-fifth of a mile a second, find approximately the number of miles which would be travelled by a space ship in a year if it travels at five times the speed of sound.
3. Suppose on a hike you cover three feet at each step. How many miles, approximately, could you cover in one million steps?
4. Using the figures given above find approximately how many times electricity could travel around the earth at the equator in one second.
5. Suppose you had the task of making a million marks on paper and you made two marks in a second. About how many hours would it take you to accomplish the task?
6. In the latter part of the 19th century, travelers in the western part of the United States traveled by covered wagon or stagecoach. Mail was sent by Pony Express. A good day's journey in a covered wagon was about 20 miles. At this rate about how long would it take to travel from New York to San Francisco, a distance of about 3000 miles?
7. Under good conditions stagecoaches could average around 60 miles per day. About how many days would you have to travel in a stagecoach to make the trip from San Francisco to New York?
8. By changing horses every ten miles and changing riders every 30 miles, the Poney Express averaged about 250 miles per day. How long approximately would it take it to make the trip above?

9. At the rate of one dollar per second, about how many days would it take to spend a billion dollars?

10. How many 8-hour days would it take to count a billion dollars at the rate of one dollar per second? How many years?

11. A commuter pays 20¢ per day for his fare on the subway. About how many days must he travel to spend a million cents?

12. Have a million days passed since the year 1? About how many years are there in a million days?

13. About how many seconds does it take for the light from the sun to reach the earth? How many hours?

14. The earth's speed in its orbit around the sun is a little less than seventy thousand miles per hour. About how far does the earth travel in its yearly journey around the sun?

15. Approximately how long would it take the space ship described in Problem 2 to reach the sun?

16. Approximately how many years would it take the space ship described in Problem 2 to reach the nearest star four light years away?

## 3-3. Small Numbers

Sometimes we have to deal with very small numbers. For these, too, scientific notation is useful. One small number is 0.000001, which we read as "one millionth". This can be written as a power of the one-tenth. To see this, make a table of the powers of one-tenth:

$$(0.1)^1 = 0.1; \quad (0.1) \times (0.1) = (0.1)^2 = 0.01$$

$$(0.1) \times (0.1) \times (0.1) = (0.1)^3 = 0.001$$

$$(0.1) \times (0.1) \times (0.1) \times (0.1) = (0.1)^4 = 0.0001$$

$$(0.1) \times (0.1) \times (0.1) \times (0.1) \times (0.1) = (0.1)^5 = 0.00001$$

$$(0.1) \times (0.1) \times (0.1) \times (0.1) \times (0.1) \times (0.1) = (0.1)^6 = 0.000001$$

Here the exponent indicates the number of times 0.1 occurs in the product. Notice that it is not the number of zeros after the decimal point but is 1 more than this number.

We know that another way to write 0.1 is $\frac{1}{10}$ and thus

$$(0.1)^6 = (\frac{1}{10})^6 = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}$$

$$= \frac{1}{10 \times 10 \times 10 \times 10 \times 10 \times 10} = \frac{1}{10^6}$$

A simpler way to write $\frac{1}{10^6}$ is $10^{-6}$ with a negative integer as an exponent. That is, we use the negative exponent $^{-6}$ to indicate that 1 is divided by the product of six tens. Similarly, 1 divided by the product of nine tens would be written $10^{-9}$. Furthermore, 1 divided by the product of four 5's would be written $5^{-4}$. Just to show how this goes we make the following table of powers of ten:

| $10^5$ | $10^4$ | $10^3$ | $10^2$ | $10^1$ | $10^0$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ |
|---|---|---|---|---|---|---|---|---|---|
| 100,000 | 10,000 | 1,000 | 100 | 10 | 1 | 0.1 | 0.01 | 0.001 | 0.0001 |

Where in each case the power of 10 occurs immediately above the number it represents. There is one entry that needs special comment. In order to make the exponents the integers in order, we define $10^0$ to be 1. In fact, it is convenient to define $2^0$ to be 1; any number, except zero, raised to the zero power we define to be 1.

Now how could we write 0.0009 in scientific notation? It is $9 \times 0.0001$, that is $9 \times 10^{-4}$. Here are some other small numbers written in scientific notations:

$$0.000004 = 4 \times 10^{-6}$$
$$0.000056 = 5.6 \times 10^{-5}$$
$$0.01234 = 1.234 \times 10^{-2}$$
$$0.12345 = 1.2345 \times 10^{-1}.$$

The numbers we have written above are large compared to

$$0.0000000000000000000000027 = 2.7 \times 10^{-23}$$

which is the approximate weight in grams of one atom of oxygen. Here the advantage of the scientific notation is most apparent.

Just as we used the scientific notation to multiply large numbers, so we can use it for products of small numbers. Suppose we want to multiply

$$0.0000057 \times 0.0000000896.$$

In scientific notation this is

$$\begin{aligned} & 5.7 \times 10^{-6} \times 8.96 \times 10^{-8} \\ = \ & 5.7 \times 8.96 \times 10^{-6} \times 10^{-8} \\ = \ & 51.072 \times 10^{-14} \\ = \ & 5.1072 \times 10^{-13} \end{aligned}$$

We can check this by the usual rule for pointing off a product. To do this we count the number of places to the right of the decimal point in 0.0000057, which is 7; this checks with the fact that the number of places to the right of the decimal point in $10^{-6}$ is 6, the number of places to the right of the decimal point in 5.7 is 1 and 6 + 1 = 7. Similarly the number of places to the right of the decimal point in 0.0000000896 is by either calculation 10 and 10 + 7 = 17 which is equal to 4 + 13, the number of places to the right of the decimal point in

$$5.1072 \times 10^{-13}.$$

In fact, we can use powers of ten to explain the rule for multiplying decimals. Suppose we are to find the product of 243.5 and 0.002. Then first we take the product as if no decimal points were present, getting 4870. Then we notice that the first member of the product has one place to the right of the decimal point and the second member has three places to the right of the decimal point. Thus, by our rule, since 1 + 3 = 4, our answer has four places to the right of the decimal point and must be written: 0.4870. The rule is:

To find the number of places to the right of the decimal point in the product of two decimals, add the number of places to the right of the decimal point in the two factors of the product.

To see why this works in terms of the example given we can write 243.5 as $2435 \times 10^{-1}$ which is one way of showing that 1 is the number of places to the right of the decimal point in 243.5; and writing 0.002 as $2 \times 10^{-3}$ which shows that 3 is the number of places to the right of the decimal point in 0.002. Then

$$243.5 \times 0.002 = 2435 \times 10^{-1} \times 2 \times 10^{-3}$$
$$= 2435 \times 2 \times 10^{-1} \times 10^{-3} = 4870 \times 10^{-4} = 0.4870$$

Since (-1) + (-3) = -4, we have the 4 which gives the number of places to the right of the decimal point in the product. It is not hard to see from this example that the rule would hold no matter what numbers we had in the product.

## Exercises 3-3

1. Write each of the following as a power of 10.
   - (a) 0.001
   - (b) 1,000,000,000
   - (c) 0.00001
   - (d) 0.00000000001
2. Write each of the following without using exponents:
   - (a) $2^4$
   - (b) $5^3$
   - (c) $2^{-3}$
   - (d) $2^{-10}$
   - (e) $2^0$
   - (f) $5^0$
   - (g) $(-1)^{10}$
3. Write each of the following in scientific notation:
   - (a) 0.093
   - (b) 0.0000786
   - (c) 0.157
   - (d) 123.56
4. Using scientific notation find the product of each of the following:
   - (a) 0.001 × 0.057
   - (b) 0.00123 × 0.000000024
   - (c) 0.0000456 × 0.00000012
   - (d) 1,000,000 × 0.00001
5. As in the text above for 243.5 and 0.002, show that the rule for placing the decimal point in a porduct holds for

   456.7 × 0.12

## 3-4. Products of Large and Small Numbers

Scientific notation is also useful in finding products of large and small numbers. Suppose we start with a simple one: 10,000,000 × 0.00001. These are powers of ten and we can write the product in the form

$$10^7 \times 10^{-5}.$$

Since $10^{-5} = \frac{1}{10^5}$ it follows that

$$10^7 \times 10^{-5} = \frac{10^7}{10^5}$$

In the numerator of the fraction we have a product of seven tens and in the denominator a product of five tens. So it can be written

$$10 \times 10 \times \frac{10}{10} \times \frac{10}{10} \times \frac{10}{10} \times \frac{10}{10} \times \frac{10}{10}.$$

But $\frac{10}{10} = 1$ and hence the product is equal to

$$10 \times 10 \times 1 \times 1 \times 1 \times 1 \times 1$$

which is $10^2$. If the product had been $10^8 \times 10^{-5}$ the answer would have been $10^3 = 1000$. So in these cases:

$$10^7 \times 10^{-5} = 10^2; \qquad 10^8 \times 10^{-5} = 10^3$$

In a similar fashion we would find that

$$10^{16} \times 10^{-9} = 10^7.$$

A quicker way to get the 7 would be to add the exponents on the left: $16 + (-9) = 7$. This, you recall, is what we did earlier for positive exponents, e.g., $10^6 \times 10^7 = 10^{13}$. What would be the answer for $10^{-7} \times 10^4$?

Now consider another product that is a little more difficult 6,000,000 × 0.000023. In scientific notation this is

$$6 \times 10^{6} \times 2.3 \times 10^{-5} = 6 \times 2.3 \times 10^{6} \times 10^{-5}$$

$$= 13.8 \times 10^{1} = 138 \text{ or, in scientific notation}$$

$$= 1.38 \times 10^{2}.$$

If we had the product 6,000 × 0.0000023, it would be

$$6 \times 10^{3} \times 2.3 \times 10^{-6} = 13.8 \times 10^{-3}$$

$$= 1.38 \times 10^{-2} = 0.0138.$$

Suppose we use what we have learned to get some idea of how small the weight in grams of an atom of oxygen really is. We wrote it as $2.7 \times 10^{-23}$ but this really does not give us much idea of its size. Here is one calculation that will help: there are about $2\frac{1}{2}$ billion people on earth. Suppose each person's weight were equal to that of one trillion atoms. Then the combined weight of these $2\frac{1}{2}$ billion persons would be

$$2.5 \times 2.7 \times 10^{-23} \times 10^{9} \times 10^{12} = 2.5 \times 2.7 \times 10^{-23} \times 10^{21}$$

$$= 2.5 \times 2.7 \times 10^{-2} = 0.067 \text{ grams, approximately.}$$

Thus the combined weight of one billion such people would be 0.027 grams, which is much less than the weight of one drop of water. (One cup of water weighs about 250 grams).

## Exercises 3-4

1. Find the values of the following products:
   (a) $10^{12} \times 10^{-3} \times 10^{15}$
   (b) $10^{12} \times 10^{-7} \times 10^{-8}$

2. Using scientific notation find the values of the following:
   (a) $2,000 \times 0.0013$
   (b) $23,000 \times 0.00024$
   (c) $2,000,000 \times 0.00000000014$
3. Multiply 3.141,592,653,589,793, the number given in the first section, by $10^{-15}$. The answer is $\pi$ to the first fifteen decimal places.
4. A gram is the weight of a cubic centimeter of water under certain physicial conditions. (A centimeter is about two-fifths of an inch). About what fraction of a cubic centimeter of water will weigh 0.03 grams?
5. A star is one billion light years from the earth. How would the number of miles this star is from the earth compare with the number of oxygen atoms it takes to weigh one gram?

## 3-5. Percent

You have already had some acquaintance with percent. You know, for instance, that "percent" means "hundredths", that is, 15 percent, written 15% is

$$\frac{15}{100} \text{ or } 15 \times \frac{1}{100} = 0.15, \text{ and}$$

$$178 \quad = \frac{178}{100} \times \frac{1}{100} = 1.78$$

If you know how to work with decimals, the only new thing about percent is translating percents into decimals and decimals into percents.

Suppose we consider a few problems involving percent. If a city's population increases from 10,000 to 10,500 in a year, it will have increased by 500 people. Now the increase of 500 people is not especially important by itself, but what is more important is the ratio of the increase to the population, that is, the fraction

$$\frac{500}{10,000}$$

which is equal to 0.05 or five hundredths. Since one hundredth is the same as one percent, we could also say that the population had increased by five percent, written 5%. If the population of the city were 1,000 instead of 10,000 and the increase were still 500 persons, it would have increased by 50% since $\frac{500}{1000} = 0.50$. If the population of the city were 100,000 instead of 10,000 and if the increase were still 500 persons, the ratio of increase would be $\frac{500}{100,000} = 0.005$ which is 0.5% or one-half of one percent. The most startling increase was for the town of 1,000 persons, since there the ratio or percent of increase was the largest.

Suppose the population of a certain town was 1000 in January of 1957, its population increased 4% during that year and 5% during the year beginning January 1958. What was its population at the end of 1958? To answer this, first notice that the first year its population increased by 4%, that is, by 0.04 × 1000 = 40 persons. Hence its population at the end of the first year was 1040. Then it increased by 5% the following year, that is by 0.05 × 1040 = 52. Thus its population at the end of the two years was 1040 + 52, that is 1092. This means that its ratio of gain over the two years was $\frac{92}{1000} = 0.092$, which is 9.2%.

The interest which a bank gives for savings accounts is also stated in terms of percent. If the bank pays 3% interest per year, the interest on \$100 for one year will be 3% of \$100. That is, the number of dollars you will receive as interest at the end of the year will be $0.03 \times 100 = 3$. And if you draw out your money plus interest at the end of the year you will have \$103. But if you leave your money in the savings account, the second year you will not only get interest for one year on the hundred dollars you originally deposited but also interest on the \$3 which you got in interest at the end of the first year. Hence the interest which your money will earn the second year will be

$$0.03 \times 103 = 3.09$$

and the total amount which you can withdraw at the end of two years would be

$$\$103 + \$3.09 = \$106.09.$$

Interest on interest computed in this fashion is called "compound interest".

Suppose we look at this another way. To get the amount at the end of the first year including interest we have

$$100 + 0.03 \times 100 = 100(1 + 0.03) = 100(1.03).$$

Thus, at three percent interest, we can find the amount at the end of a year by multiplying the amount at the beginning of that year by 1.03. Hence in order to get the amount at the end of the second year we multiply the 100(1.03), which we have at the beginning of the second year, by 1.03 to get

$$100(1.03) \times (1.03).$$

A shorter way of writing this would be $100(1.03)^2$. If you multiply this out you will get 106.09 as before.

Notice that we could also have used this system for computing the population of the city in the previous example after an increase of 4%, that is,

$$1000 \times 1.04 = 1040,$$

and the population at the end of the two years would be

$$1000 \times 1.04 \times 1.05.$$

## Exercises 3-5

1. Express the following decimals as percents:

   0.12, 0.03, 1.53, 0.002

2. Express the following percents as decimals:

   15%, 165%, 15.2%, 0.2%

3. A city of 1,000,000 population gains 50,000 in a year. What is its percent of increase?
4. If a bank pays 4% interest and you put $1000 in at the beginning of the year, how much can you draw out at the end?
5. A vacuum cleaner salesman gets a commission of 7%. That is, he gets 7% of the price of each vacuum cleaner which he sells. How much would his commission be on a cleaner which he sold for $75?
6. A real estate salesman gets a commission of 5% on every house which he sells. What would be the amount of the salesman's commission on a house which he sells for $25,000?

7. If you put $1000 in a savings account and left it there three years and if the bank pays 3% interest per year, what amount could you draw out at the end of that time?

8. Suppose a bank pays $1\frac{1}{2}$% interest every six months on a savings account, is this better or worse than paying 3% every year? Show why your answer is correct.

9. Suppose in the case of the town of 1000 population mentioned above, the population increased by 5% the first year and 4% the second. Do you think the population at the end of two years would be more, less, or the same as the 1092 which we obtained above? What are the reasons for your answer? Then work it out and see if you are right.

10. In a certain state, the income tax is 1% on the first $1000 of net income, 2% on the second $1000, 3% on the third $1000, and so on up to 10%. That is, if the net income were $2,357 a man would pay:

| | | | |
|---|---|---|---|
| 1% of first | $1000 | or | $10 |
| 2% of second | $1000 | or | $20 |
| 3% of remaining | $357 | or | $10.71 |
| His total tax would be: | | | $40.71 |

How much income tax would a man pay on a net income of $4735?

11. Real estate taxes are usually stated in "mills" that is, in thousandths. For instance, a tax rate of 42 mills would be 0.042 or 4.2%. If the tax rate is 42 mills, how much tax would a man pay on a $15,000 house?

12. A city government estimates that the total value of the taxable property within the city limits is about \$120,000,000. They wish to set the taxes so that their income from that source will be about \$2,760,000. What should the tax rate in mills be? (See Problem 11.)

13. The "average" of a team in a league is the decimal (to three places) which represents the ratio of its games won to its games played. That is, if a team played 11 games and won 3, its "average" would be $\frac{3}{11}$ which, to three decimal places, is 0.273. What would its overall average be if it played five more games, three of which it won?

14. A team wins 15 games and loses 25, what is its average?

15. A team plays 60 games in a season. Its average after playing 25 games is .400. If it won the remaining 35 games, what was its average for the season?

16. Suppose the rival team to that in Problem 15 had an average of .800 after playing 25 games. If its average for the remaining 35 games was .400, what would be its average for the entire 60 games?

17. There are four teams A, B, C, D in a league. The games of the season are represented by the following table:

| | A | B | C | D |
|---|---|---|---|---|
| A | 0 | 8 | 4 | 1 |
| B | 12 | 0 | 7 | 8 |
| C | 16 | 13 | 0 | 11 |
| D | 19 | 12 | 9 | 0 |

where the table means that A won 8 games with B, 4 with C and 1 with D; while B won 12 games with A, 7 with C and 8 with D; etc. What is the average of each team?

*18. A baseball league has ten teams and each team plays the same number of games during the season with every other team. What would be the approximate sum of the averages of the ten teams at the end of the season?

19. Call the "wholesale price" of an article the amount the store pays for it and the "retail price" the price the store sells it for. If the store sets the retail price 5% above the wholesale price, what will be the retail price of an article whose wholesale price is $75? If the store gives the clerk at the counter a commission of 2% of the retail price, what is his commission on the same article? What will be the profit which the store makes after paying the commission? What percentage will this be of the wholesale price?

20. Dick knew he could buy a motor scooter for $200 and wanted to sell his jalopy for enough to give him this amount. Not being able to sell it himself he approached a salesman who said he would charge a 10% commission. So Dick figured that 10% of $200 is $20 and therefore told the salesman to sell it for $200. Did this accomplish what he wanted?

21. An owner wished to get $24,000 for his house after the 5% commission of the real estate salesman had been paid. He found that 5% of $24,000 is $1,200. So he set the price at $25,200. Did this accomplish what he wanted?

## 3-6. Discount

Suppose we look first at the results of Problem 20, Section 3-5. We see that 10% of \$220 is \$22 and by the time the commission is paid, Dick would be \$2 short of the \$200 he needed for the scooter. He should have directed the salesman to sell it for just a little more, for instance, \$225. Then the commission would have been \$22.50 and the net amount he would have received would have been \$202.50. But of course he need not have sold it for quite that much; so he could have tried a smaller amount; and so on. But he could have found an exact figure. He could have let P stand for the price he was to set and then the price he would get after the commission would be:

$$P - 0.10 \times P = 0.9 \times P.$$

Thus he wants to set P so that

$$0.9 \times P = 200,$$

that is,

$$P = \frac{200}{0.9} = \$222.22.$$

For this price the commission is \$22.22 and he would have exactly \$200 left after paying the commission.

We have shown that Dick should have set a price of \$222.22 for his car. Now if a friend had come along and wanted to buy, Dick couldhhave told him he would take 10% off the price since he would save the fee of the salesman. In other words he would offer his friend a "10% discount", which is another way of saying that he would sell it to him for 10% less than the price posted. Goods at a store are often sold at a discount.

Similarly, if one were to seek a loan of $1000 from a bank which charged interest at 6%, the bank official might say, "the interest on $1000 is $60. So I will subtract this from the $1000 and give you $940. Then at the end of the year, you pay us back $1000." If it were done this way, the bank would be lending money at a "discount rate of 6%"--not an interest rate of 6%.

Notice that $60, being 6% of $1000, is more than 6% of $940. In fact, $60 is almost 6.4% of $940. The discount rate of 6% is approximately the same as an interest rate of 6.4%.

## Exercises 3-6

1. A $4000 car is sold at a discount of 7%. For what price was it sold?
2. A town having 1000 population decreased 5% during a year. What was its population at the end of the year?
3. In Problem 21 of Section 3-5, what should have been the price which the owner set on his house?
4. A bank makes a loan at a discount rate of 5%. What would be the equivalent rate of interest? (Hint: Try this first for a loan of $100, second for $1000. Then try to draw a general conclusion.)
5. A town decreased in population by 5% one year and increased by 5% the next year. Would its population at the end of the two years be less, than, more than, or the same as that at the beginning? Explain your answer.

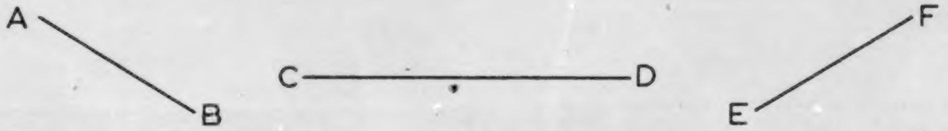
6. A store is going out of business and the first week all goods are sold at a discount of 25% (that is, marked down 25%). The second week they are sold at a discount of 10% on the price for the first week. Is the total discount in the end 35%? Give reasons for your answer.

## Unit 4

# CONGRUENCE AND THE PYTHAGOREAN PROPERTY

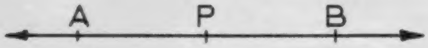
### 4-1. Basic Constructions

The ruler and compass are the tools used in the construction of geometric figures. The ruler is used as a straight edge to draw a representation of a straight line. The compass is used to draw a representation of a circle, or part of a circle called an arc. In geometry we often compare two figures by comparing their shapes and comparing their sizes. When two figures have the same shape and the same size, we may say that they are congruent to one another. As a very simple case, consider two line segments. Appearance tells us that any two segments have the same shape: being parts of lines, they are both straight; and being segments, both of them include their endpoints. Now the size of a segment is its length. To say that two segments have the same size means that they have the same length. Thus we now have a new way to refer to two segments with the same length; we may say they are congruent segments.



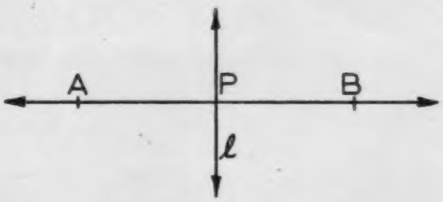
In terms of a unit of length, $\overline{AB}$ and $\overline{EF}$, as shown, have equal measure. We may say that $\overline{AB}$ is congruent to $\overline{EF}$ and we write $\overline{AB} \cong \overline{EF}$. On the other hand, $\overline{CD}$ is not congruent to $\overline{EF}$, since $\overline{CD}$ is longer than $\overline{EF}$. You will recall that we have learned another way to write in symbols that $\overline{AB}$ and $\overline{EF}$ have the same length; we write that their measures are equal: $AB = EF$.

The terms "right angle" and "perpendicular" lines should be familiar to you. You will recall that a right angle is an angle whose measure in degrees is 90. The two lines that form the right angle are perpendicular lines. A line segment is bisected at a point if the point separates it into two segments with equal measure.



Point P bisects $\overline{AB}$ if $\overline{AP} \cong \overline{PB}$.

Suppose that segment $\overline{AB}$ is bisected at point P by a line that is perpendicular to $\overline{AB}$, such as line $\ell$. Line $\ell$ is said to be the *perpendicular bisector* of $\overline{AB}$.



*Definition*. The *perpendicular bisector* of a line segment is a line that passes through the midpoint of the segment and forms right angles with the segment.

Let us see how to use the compass and ruler to construct the representation of the perpendicular bisector of segment $\overline{CD}$. In order to construct such a line, it is necessary to locate two points on it. Since a line has no breadth or thickness, it cannot be seen. In our construction work we will always be drawing *representations* of lines and circles.

Set the compass so that the distance between the two points of the compass is more than half of $\overline{CD}$. Place the sharp point of

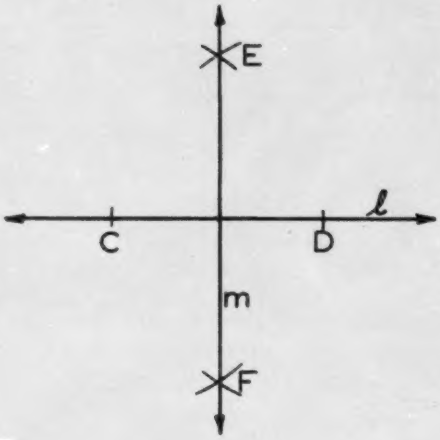


Figure 4-1a

the compass on C and make an arc above $\overline{CD}$ and another arc below $\overline{CD}$. Keep the set of the compass the same, place the point on D and make two arcs that intersect those drawn from C. Label the points of intersection of the arcs E and F. With a ruler draw line m through E and F. Line m is the perpendicular bisector of $\overline{CD}$. If the construction is accurate, $\angle$ COE, $\angle$ EOD, $\angle$ COF, $\angle$ FOD should all measure 90 degrees on your protractor. Also, segment $\overline{CO}$ equals segment $\overline{OD}$. Could you check this with your compass? Try it.

Sometimes it is necessary to construct a line that is perpendicular to a given line and also passes through a given point.

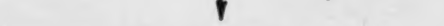


Figure 4-1b

If we wish to construct a perpendicular to line $\ell$ through point P, we need to find one more point on the second line. With P as center and a convenient radius, draw two arcs that intersect line $\ell$ in points G and H. Then with G as a center and a radius longer than $\overline{GP}$, draw an arc above (or below) line $\ell$. With the same radius with H as a center, draw an arc that intersects the arc drawn from G at J. With a ruler draw a line through J and P. This line is perpendicular to line $\ell$ at point P.

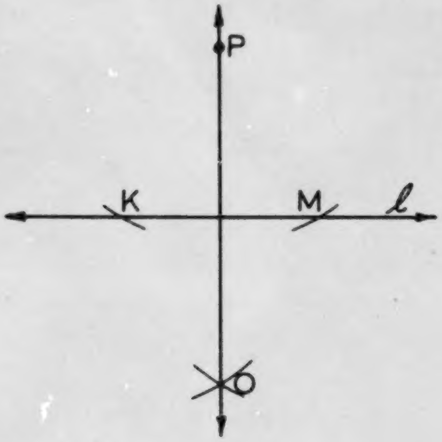
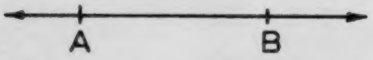


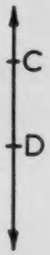
Figure 4-1c

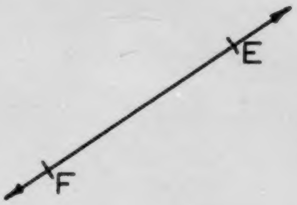
In case the given point P is not on the given line $\ell$, the construction is basically the same as when P is on the given line. With P as a center and a convenient radius, draw two arcs that intersect line $\ell$ in points K and M. With K and M as centers, draw two arcs that intersect at O (use the same radius for these two arcs). The line through P and O is perpendicular to line $\ell$.

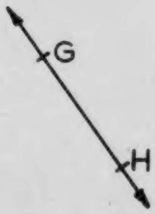
Exercises 4-1a

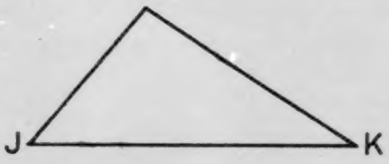
1. Draw line segments of approximately the same length and in approximately the same position as shown below. Construct the perpendicular bisector of each segment. Do not erase any of the arcs used in the construction.



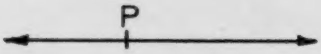


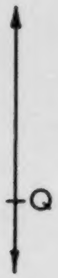


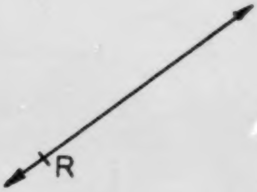


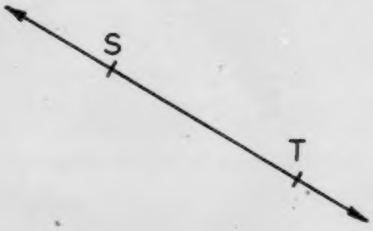


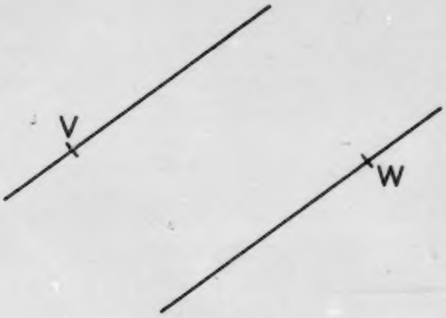
2. Construct a perpendicular to a line at the labeled point on the line. Remember that a line extends indefinitely in two directions. Place the given lines and points on your paper in approximately the same position.







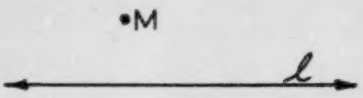


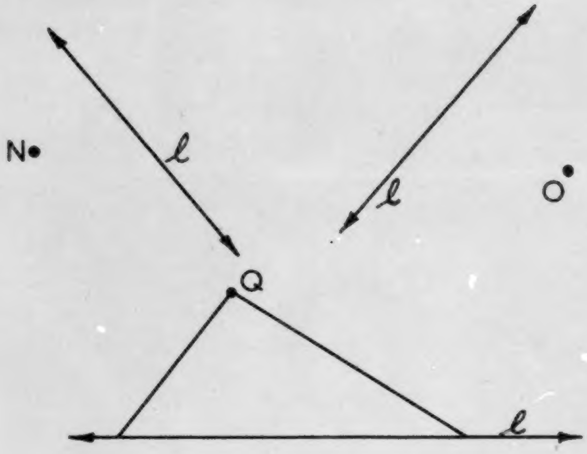


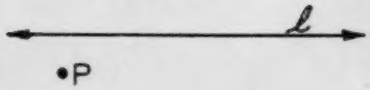
(f) What do you notice about the two lines constructed at points S and T in part (d)?

(g) What seems to be true about the two given lines and the two perpendiculars constructed in part (e)?

3. Construct a perpendicular to a line $\ell$, through a point not on the line. Place the given lines and points on your paper in approximately the same position.







4. Construct a rectangle whose sides are 1" and 2" long. Draw the 2" side as a segment on a line. Construct a perpendicular at each end of the segment. You can decide on the remaining steps in the construction.

5. Construct a square whose sides are 3 centimeters long.

*6. Try to construct a rectangle whose sides are 20 millimeters and 45 millimeters long with only one construction of perpendicular lines.

Constructions with angles.

We will discuss two constructions that are concerned with angles of any measure. Let us review the terms that apply to angles. An angle is a set of points consisting of two rays with an end point in common, and not both on the same straight line. An angle has an interior and an exterior. In the figure, the exterior of the angle is shaded. Another name for the rays of

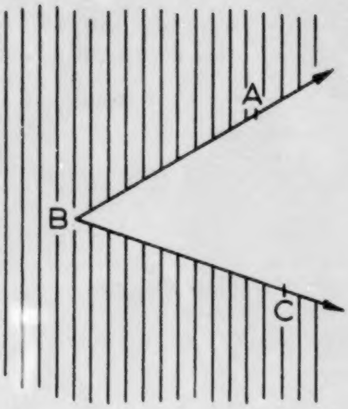
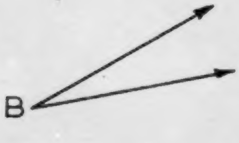
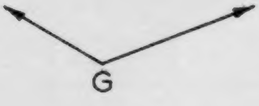


Figure 4-1d

the angle is "sides" of the angle. The common end point of the rays of an angle is called the vertex of the angle. The measure of an angle is the measure of the interior of the angle. The unit for such measure is the protractor. An angle may be named by 3 letters, one at the vertex (middle letter), and one at a point on each ray. The symbol $\angle$ stands for the word "angle". In the figure the angle may be named $\angle A\underline{B}C$ or $\angle C\underline{B}A$ . If no confusion results, we may name an angle by the vertex letter as $\angle$ B.

Since an angle is a geometric figure, we may speak of one angle being congruent to another. What do you think this would mean? The general notion of congruence tells us that congruent angles have the same size and same shape. Now the size of an angle is indicated by its measure. Hence, to say that two angles have the same size means that their measures (in degrees) are the same. Appearance tells us that any two angles with equal measures are shaped alike. Thus we now have a new way to refer to two angles with equal measure: we may say that they are congruent angles.





The interiors of angles B and F, as shown, have equal measure. We may say $\angle B$ is congruent to $\angle F$, and we may write $\angle B \cong \angle F$. Recall that another way of expressing this relationship is: $m(\angle B) = m(\angle F)$. In any given situation, our choice of expression will depend upon whether we wish to emphasize the angles as being congruent figures or the measures as being equal numbers. Which pairs of angles shown above are not congruent?

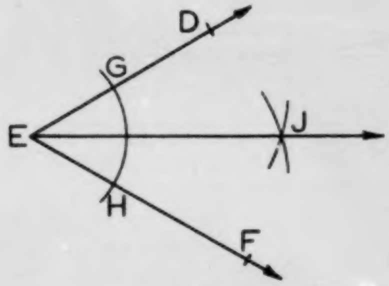


Figure 4-1e

To bisect an angle is to construct another ray through the vertex, interior to the angle, so that the two angles formed by this ray and the sides of the original angle are congruent.

To bisect $\angle DEF$, draw an arc with E as a center that intersects $\overrightarrow{ED}$ in point G, and $\overrightarrow{EF}$ in point H. With the compass set at one distance, draw two arcs from points F and H that intersect at point J. A line through E and J bisects $\angle DEF$. Use your protractor to measure $\angle DEF$, $\angle DEJ$, and $\angle JEH$.

An angle whose measure in degrees is less than 90 is called an acute angle. An angle whose measure in degrees is more than 90 is called an obtuse angle. We have constructed right angles

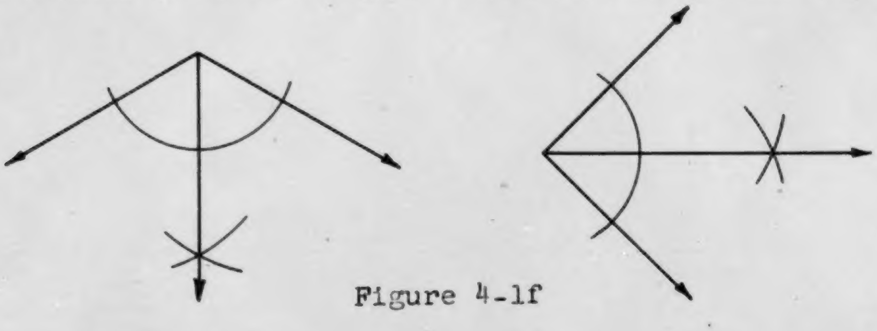


Figure 4-1f

with perpendicular lines. The figures show an obtuse angle that has been bisected, and a right angle that has been bisected. Measure all the angles with your protractor.

Suppose we wish to construct an angle congruent to a given angle.

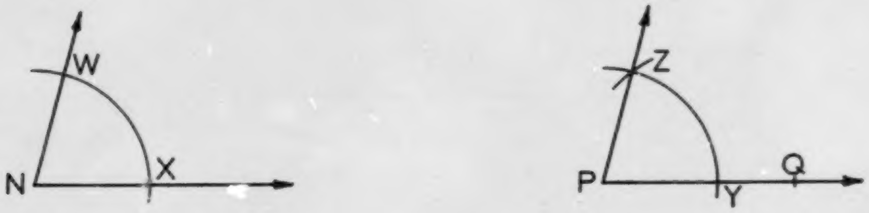
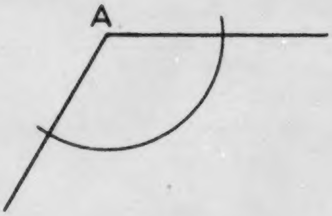


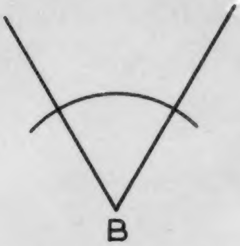
Figure 4-1g

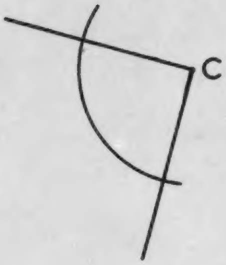
Given ∠ N and $\overrightarrow{PQ}$. We wish to construct another ray from point P so that it will make an angle with PQ that is congruent to ∠ N. With a convenient radius draw an arc with N as a center that intersects the sides of ∠ N in points X and W. With the same radius draw an arc with P as a center that intersects $\overrightarrow{PQ}$ in point Y. Change the set of the compass to the distance WX. With the compass set at this distance, make an arc with Y as a center that intersects the arc drawn from P at point Z. A ray drawn from P through Z completes the angle. Measure ∠ WNX and ∠ ZPY with your protractor. Are the measures equal?

## Exercises 4-1b

1.







(a) Starting with a ray drawn in any convenient position, construct an angle congruent to the angle in the figure (a). You must set your compass carefully to the exact distance used in the figure.

(b) Repeat with figure (b).

(c) Repeat with figure (c).

2. (a) With your protractor measure $\angle$ A and also the angle that you constructed.

(b) Repeat the two measurements with $\angle$ B and the angle that you constructed.

(c) Repeat with $\angle$ C.

3. (a) Draw an acute angle and bisect it.

(b) Draw an obtuse angle and bisect it.

(c) <ins>Construct</ins> a right angle and bisect it.

4. Using your protractor, find the measure in degrees of the 9 angles drawn and constructed in problem 3.

5. Using the symbols =, >, <, make a definite statement about the measure of:

(a) Each angle formed by bisecting a right angle.

(b) Each angle formed by bisecting an acute angle.

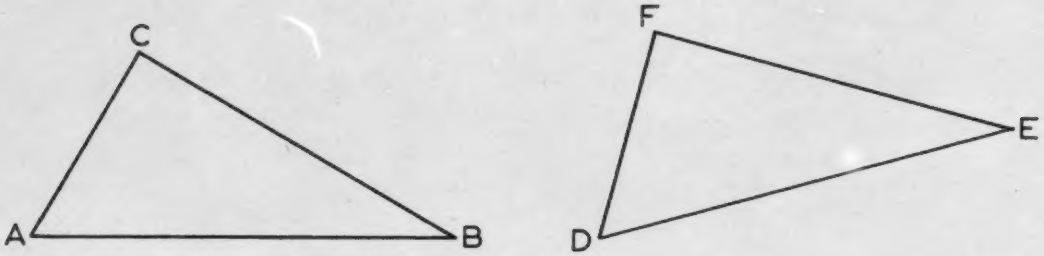
(c) Each angle formed by bisecting an obtuse angle.

## 4-2. Congruent Triangles

When leaves are drifting to the ground in the fall, have you ever tried to find two leaves from the same tree which have exactly the same size and shape? While we can tell whether a leaf is from an oak tree or a maple tree or some other kind of a tree, it is unusual to find two oak leaves which seem to be exactly the same size and shape if we place one on top of the other to compare them. You may also have wondered why, among the faces of all the people you know, or even among all the people in a great city, you seldom find two which are very much alike.

The wheat fields of two farmers may be "exactly" the same shape and size, although they usually are not. A manufacturer of large bolts of a certain kind employs methods of "quality control" to try to make sure that the bolts he makes will be nearly alike as possible. Mathematics is used to determine whether objects that appear to be alike actually are, and to determine the amounts and importance of difference in the objects.

In order to study the characteristics of objects that are exactly alike, we begin, as scientists usually do, with a very simple situation. It turns out that the case of two triangles is the key to the general problem of determining when two objects have exactly the same size and shape. Remember, a polygon is a closed curve composed of line segments; and a triangle is a polygon with three sides.



The triangles ABC and DEF appear to have the same size and shape. If triangle DEF were traced on paper and the paper cut along the sides of the triangle, the paper model would represent a triangle and its interior. The paper model could be placed on triangle ABC so that the two triangles would exactly "fit". If point D were placed on point A with $\overrightarrow{DF}$ along $\overrightarrow{AC}$, point F could fall on point C, and point E could fall on point B. In these two triangles there would be the following pairs of congruent segments and congruent angles.

| | |
|---|---|
| AB ≅ DE | ∠ ABC ≅ ∠ DEF |
| CB ≅ EF | ∠ BCA ≅ ∠ EFD |
| CA ≅ FD | ∠ CAB ≅ ∠ FDE |

Use your ruler and protractor to check these measures. Notice that we have talked about the three sides of a triangle and the three angles formed by the three sides. Two triangles such as triangle ABC and triangle DEF that have the same size and shape are called congruent triangles. The symbol ≅ stands for the word "congruent".

## Exercises 4-2a

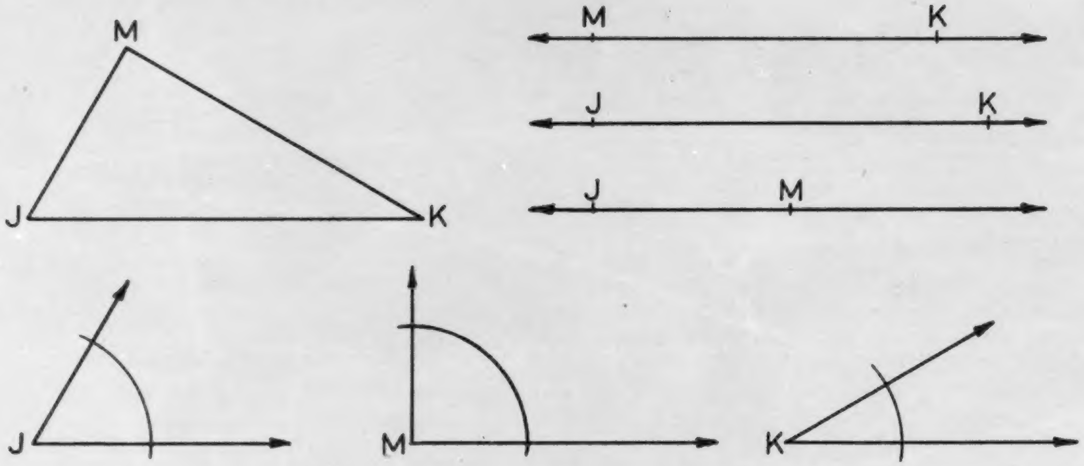


Figure 4-2a

In Figure 4-2a the segments on the right are the same length as the corresponding sides of triangle JKM; and the angles below are congruent to the corresponding angles of triangle JKM.

1. With your compass and ruler construct a triangle as follows.
   (a) Set compass for the length $\overline{JK}$, and mark along a line. Use segment $\overline{JK}$ for one side of the triangle: Construct $\angle M$ at one end of $\overline{JK}$; construct segment $\overline{JM}$ as the second side of $\angle M$. Draw the third side of the triangle through the end points marked on the two sides of $\angle M$.
   (b) Is your triangle the same size and shape as triangle JKM? If not, how is it different? Use compass and ruler to answer.
2. Construct a triangle as follows:
   (a) Use segment $\overline{JK}$ for one side. Construct $\angle M$ at point J

with $\overline{JK}$ for one side of the angle; construct $\angle J$ at point K with $\overline{KJ}$ for the one side of the angle. Extend the sides of the two constructed angles until they intersect.

(b) Is your triangle the same size and shape as triangle JKM? If not, how is it different?

3. Construct a triangle as follows:

(a) Use segment $\overline{JK}$ for one side, with $\angle J$ constructed at point J and $\angle K$ constructed at point K with $\overline{JK}$ a side of each angle.

(b) Is your triangle the same size and shape as triangle JKM? Use compass and ruler to check.

4. Construct a triangle as follows:

(a) Use $\overline{JK}$ for one side; construct $\angle J$ at point J; $\overline{JM}$ marked along the second side of the angle; draw side $\overline{MK}$ through the points M and K.

(b) Is your triangle the same size and shape as triangle JKM? Use compass and ruler to check.

5. Construct a triangle as follows:

(a) Use $\overline{JK}$ as one side; set compass for the length $\overline{JM}$ and with J as a center make an arc above $\overline{JK}$; set compass for the length of $\overline{MK}$ and with K as a center make an arc that intersects the first arc. Connect J with the intersection of the arcs by a straight line; connect K with the intersection of the arcs.

(b) Is your triangle the same size and shape as triangle JKM? Use compass and ruler to check.

In exercise 4-2a three constructions resulted in a triangle congruent to the given triangle JKM. In one figure the congruent triangle was constructed by using side $\overline{JM}$, $\angle J$, and side $\overline{JK}$. What do you notice about the position of the angle in relation to the two sides? This arrangement of two sides and an angle is called "two sides and the included angle". Another such group would be side $\overline{JK}$, $\angle K$, and side $\overline{KM}$. Is there another such group?

Property 1. Two triangles are congruent if two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other triangle.

In another figure the congruent triangle was constructed by using $\angle J$, side $\overline{JK}$, and $\angle K$. What do you notice about the position of the side in relation to the two angles? This arrangement of two angles and a side of a triangle is called "two angles and the included side". Another such group would be $\angle K$, side $\overline{KM}$, and $\angle M$. Is there another such group?

Property 2. Two triangles are congruent if two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other triangle.

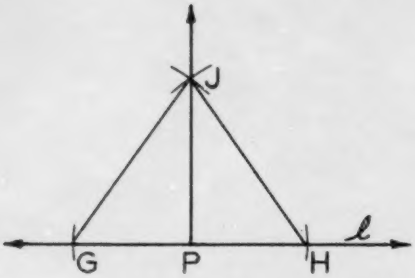
In a third figure the congruent triangle was constructed by using side $\overline{JM}$, side $\overline{JK}$, and side $\overline{KM}$.

Property 3. Two triangles are congruent if the three sides of one triangle are congruent respectively to the three sides of the other triangle.

In three figures of a triangle congruent to triangle JKM each angle was congruent to the corresponding angle in the given triangle, and each side was congruent to the corresponding side of the given triangle.

Property 4. If two triangles are congruent, then each pair of corresponding angles is congruent and each pair of corresponding sides is congruent.

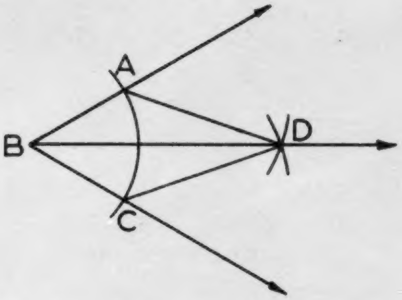
In the construction of a perpendicular to a line through a given point in the line, we used two of the properties about congruent triangles.



In the first construction we show two arcs intersecting at point J. If we draw the segments $\overline{GJ}$ and $\overline{JH}$, then there are two triangles, GPJ and PHJ formed. By construction GP ≅ PH, GH ≅ JH. The perpendicular to line ℓ has the segment JP on it. JP is a side of each triangle, and so is called a common side. Sometimes we say JP ≅ JP to indicate a common side. By applying Property 3, we know that triangle GPJ ≅ triangle PHJ, since we have three sides of one triangle congruent to three sides of the other triangle. Angle JPG (opposite $\overline{GJ}$) in triangle GPJ corresponds to angle JPH (opposite $\overline{JH}$) in triangle PHJ. By applying Property 4, we know that ∠ JPG is congruent to ∠ JPH. The two rays, $\overrightarrow{PG}$ and $\overrightarrow{PH}$ represent an angle measure of 180 degrees. Hence, angles JPG and JPH each measure 90 degrees and are right angles.

## Exercise 4-2b

1. In the figure the construction of the bisector of ∠ ABC is shown. Two segments, $\overline{AD}$ and $\overline{DC}$ are drawn.



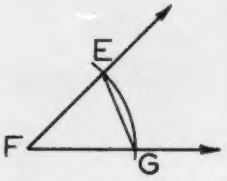
   (a) What parts of triangle ABD are congruent to corresponding parts of triangle BCD by construction?

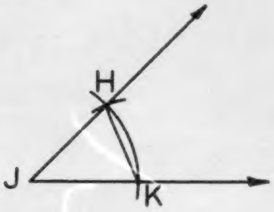
   (b) Is there another part of triangle ABD that must be congruent to a part of triangle BCD? Why?

   (c) Is triangle ABD congruent to triangle BCD? Why?

   (d) Is ∠ ABD congruent to ∠ DBC? Why?

2. In the figure the construction of ∠ HJK congruent to ∠ EFG is shown. Segments $\overline{EG}$ and $\overline{HK}$ are drawn.



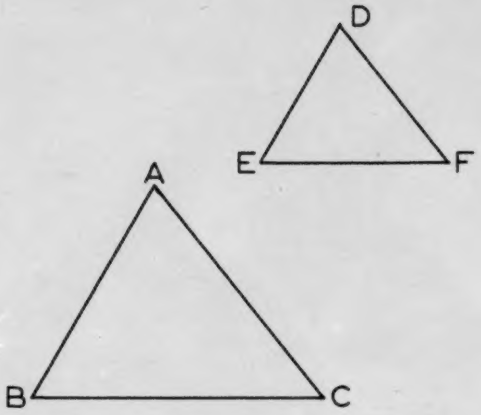


   (a) What parts of triangle EFG are congruent to corresponding parts of triangle HJK by construction?

   (b) Is triangle EFG congruent to triangle HJK? Why?

   (c) Is ∠ J congruent to ∠ F? Why?

3. Use your protractor to find the measure in degrees of the 3 angles in each triangle.

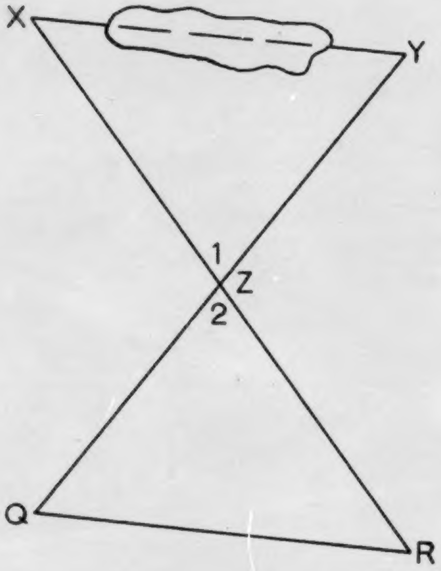


   (a) Are there some pairs of congruent angles. If so, list them.

   (b) Could we say that the triangles are congruent? Why?

   (c) Suppose that triangle DEF were constructed with the same size angles, but with $\overline{EF}$ the same length as $\overline{BC}$. What would be true about the two triangles? Why?

4. Mr. Thompson wishes to measure the distance between two posts on edges of his property. A grove of trees between the two posts (X and Y) make it impossible to measure the distance $\overline{XY}$ directly. Mr. Thompson locates point Z such that he can lay out a line from X to Z and continue it as far as needed. Point Z is also in a position such that Mr. Thompson can lay out a line $\overline{YZ}$ and continue it as far as needed. Mr. Thompson knows that $\angle 1 \cong \angle 2$ since they are vertical angles. He extends $\overline{ZY}$ so that $\overline{QZ} \cong \overline{YZ}$.

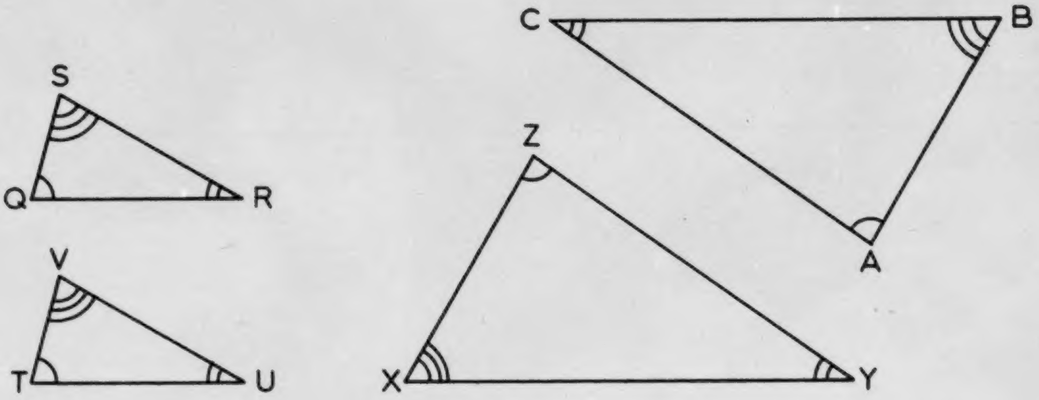


(a) How does he locate point R?

(b) What property does Mr. Thompson apply in locating points Q and R?

(c) How does Mr. Thompson determine the length of $\overline{XY}$? Why?

5. Corresponding sides of congruent triangles are sides that are opposite pairs of congruent angles.



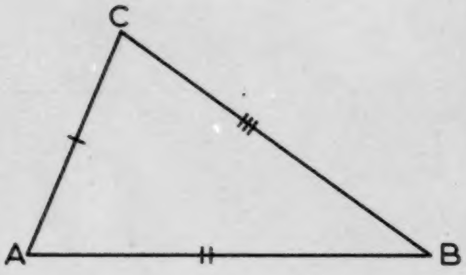
In the congruent triangles, QRS and TUV, the pairs of congruent angles are marked. That is, $\angle Q \cong \angle T$, $\angle R \cong \angle U$, $\angle S \cong \angle V$.

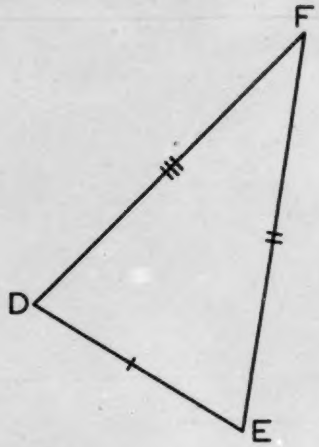
(a) Make a statement of congruence about each pair of corresponding sides.

(b) Congruent triangles XYZ and ABC have the pairs of congruent angles marked. Make a statement of congruence about each pair of corresponding sides.

(c) Repeat (b) with the figure in problem 4.

6. Corresponding angles of congruent triangles are angles that are opposite congruent sides.

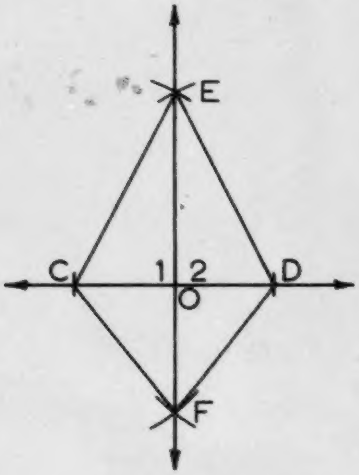




(a) In the congruent triangles, ABC and DEF, the pairs of congruent sides are marked. That is, CA $\cong$ DE, AB $\cong$ EF, BC $\cong$ DF. Make a statement of congruence about each pair of corresponding angles.

(b) Make a statement of congruence about each pair of corresponding angles in triangles XYZ and ZQR in problem 4.

*7.



The construction of the perpendicular bisector of segment $\overline{CD}$ is shown. Usually the same length radius is used for the four arcs. However, it is only necessary for the two arcs that intersect on one side of the segment to have the equal radii. Thus, the arcs drawn from C and D that intersect at E have equal radii, and the two arcs drawn from C and D that intersect at F have equal radii. By applying some of the properties about congruent triangles, show why $\overline{EF}$ bisects $\overline{CD}$ and is perpendicular to $\overline{CD}$. Hint. First think about the large triangles CFE and EDF, then about another pair of triangles that seem to be congruent.

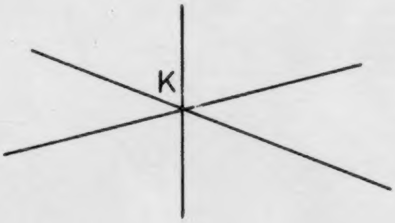
*8. (a) How many pairs of congruent triangles are there in the figure for problem 7? List them by pairs.

(b) Show by appropriate marks the corresponding sides of congruent triangles.

(c) List the pairs of corresponding angles of congruent triangles that are congruent.

## 4-3. Concurrent Lines

An interesting study of triangles deals with concurrent lines. Three or more lines on a point are said to be concurrent lines. The figure shows three concurrent lines on the point K.



Several sets of concurrent lines are associated with triangles. Using your straightedge and compass, carefully construct a triangle and the perpendicular bisectors of its sides. Do the perpendicular bisectors seem to meet in a point? If you did your work carefully, you found that the three perpendicular bisectors met in a point. Does this prove that the perpendicular bisectors of every triangle meet in a point? You have already found in your study of mathematics that one special case does not prove a problem in general. It is true, however, that the 3 perpendicular bisectors do meet in a point in every triangle. If you continue your study of mathematics you will one day prove this theorem. The figure you have just constructed should be similar to the figure below.

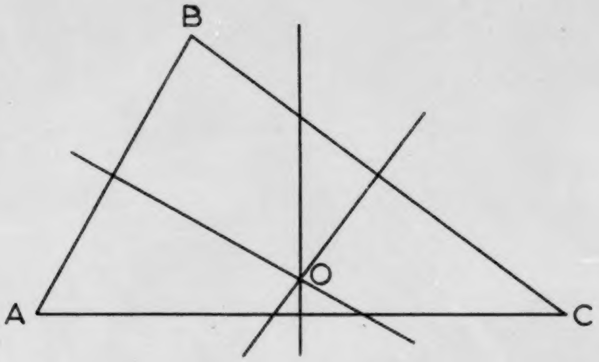


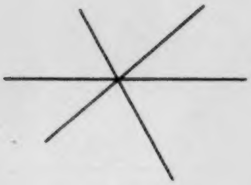
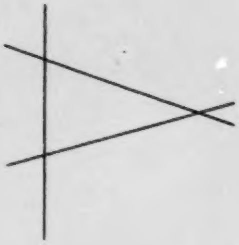
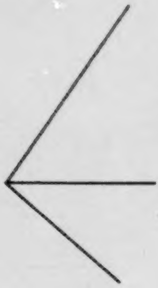
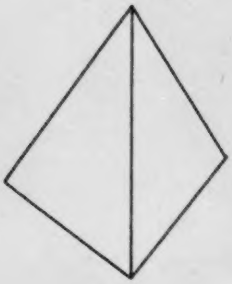
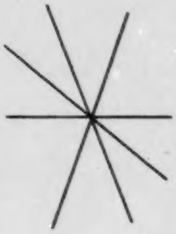
Figure 4-3

Do you notice any relationship between a circle with center at O and the vertices of the triangle? Use your compass and construct a circle with center at O and a radius $\overline{OA}$. What seems to be true?

## Exercises 4-3

1. Draw the following:
    - (a) 3 concurrent lines
    - (b) 4 concurrent lines
    - (c) 5 concurrent lines
2. Draw three concurrent rays such that the endpoint of the rays is the only element in the intersection set of the three rays.
3. How many angles are formed by the rays in exercise 2?

4. Which of the following appear to be concurrent lines or rays? In each figure consider all lines shown.

5. Using your straightedge and compass, bisect the sides of a triangle. Now construct each line which connects a vertex of a triangle with the mid-point of the opposite side. Do these lines seem to meet in a point? These lines are called the "medians" of a triangle.

6. Using your straightedge and compass, construct the line from each vertex of a given triangle which is perpendicular to the opposite side. Do these lines seem to meet in a point? These lines are called the "altitudes" of the triangle.

7. Using your straightedge and compass, construct the bisector of each angle of a given triangle. Do these lines seem to meet in a point?

Note: In working problems 5, 6, and 7 you have discovered three more important sets on concurrent lines. Of course, your special cases do not prove that the three sets are in general concurrent. Proofs are possible; however, we will not consider them.

## 4-4. Quadrilaterals

We have been calling a polygon with 3 vertices joined by 3 line segments a triangle. Do you know what the prefix "tri" means? Think of words like "triangle", "trio", "triple", and the like. Does this suggest the meaning of the word triangle? In a similar sense we call a polygon with 4 sides a quadrilateral. If R, S, T, and V are 4 distinct points, no three of which are on one line, then the polygon (See Figure 4-4a) formed by the segments $\overline{RS}$, $\overline{ST}$, $\overline{TV}$, and $\overline{VR}$ is a quadrilateral, and is referred to as the quadrilateral RSTV. The 4 segments are its isdes, and the points R, S, T, and V are called its vertices. The angles RST, STV, TVR, and VRS (also called "angles S, T, V and R") are the angles of the quadrilateral.

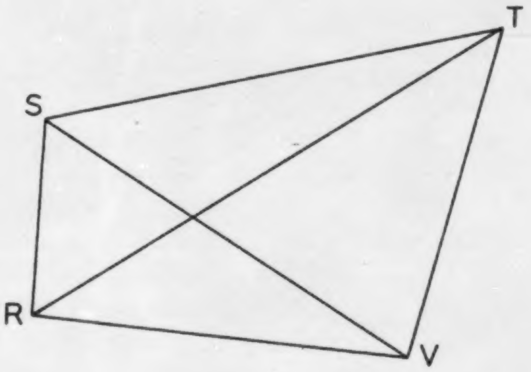


Figure 4-4a

Since we have just made a study of triangles, we are now ready to consider polygons of 4 sides, or the quadrilateral. Certain special kinds of quadrilaterals are not only interesting, but very important.

Before considering some special cases, let us consider Figure 4-4a again. The line segments joining opposite vertices are called the diagonals of the quadrilateral. $\overline{SV}$ and $\overline{RT}$ are the two diagonals. You remember that the sum of the measures of the angles of a triangle is $180^\circ$. What do you think would be the sum of the measure of the angles of a quadrilateral? Look at Figure 4-4a again. Into what kind of polygons does $\overline{SV}$ divide the quadrilateral? Can you now tell the sum of the measures of the angles R, S, T, and V? The unit of measure for angles that we shall use is the degree.

## Parallel Lines

You have already studied parallel lines. Before considering the parallelogram, let us study parallel lines for a few minutes by way of review and also add some new ideas.

The opposite sides of your straightedge may be considered to be parallel. Use it to draw two parallel line segments $\overline{MN}$ and $\overline{RS}$, similar to those shown in Figure 4-4b. Now draw any line $\overleftrightarrow{CD}$ intersecting $\overline{MN}$ at A and $\overline{RS}$ at B. Using your protractor find the

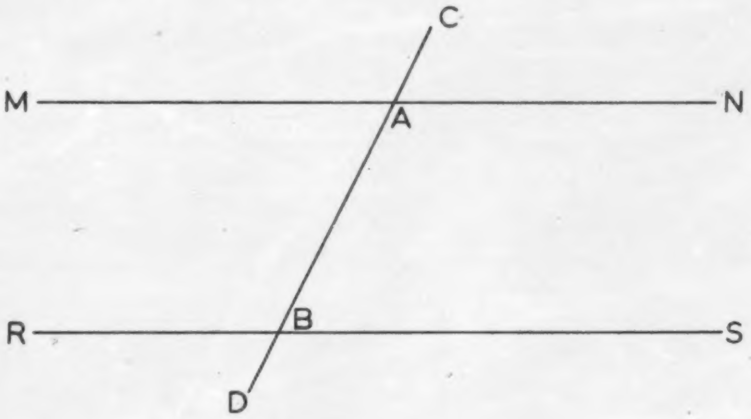


Figure 4-4b

m ($\angle$ NAB) and the m ($\angle$ SBA). If the opposite sides of your straightedge are really parallel and you did your work carefully, you should find the sum of these two measures to be 180. $\overleftrightarrow{CD}$ is called a transversal. When a transversal cuts two parallel lines the angles formed, having the position of $\angle$ NAB and $\angle$ SBA, have a total measure of 180. The converse is also true. That is, if any two lines are cut by a transversal, and the sum of the measures of the angles in the positions corresponding to $\angle$ NAB and $\angle$ SBA is 180, then the lines are parallel.

Let us consider Figure 4-4b again. This time measure $\angle$ MAB and $\angle$ RBA. Are these angles congruent? They are if $\overline{MN}$ and $\overline{RS}$ are parallel and you did your work carefully. It is also true that if two lines are cut by a transversal and the angles in positions corresponding to $\angle$ MAB and $\angle$ RBA are congruent, then the lines are parallel. Are there other pairs of angles which are congruent or the sum of whose measures is 180? See if you can find some.

Parallelogram

A parallelogram is defined as a quadrilateral having its opposite sides parallel. It is also true that if the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Using your ruler and protractor make the following drawing. Select any point on your paper and call it A. Now draw an angle at A having the measure of 60. Next, mark off on one ray $\overline{AB}$ 2 inches long, and on the other ray $\overline{AD}$ 3 inches long. Make an angle at B, with $\overline{AE}$ as one side, having a measure of 120. Locate a point C on the other side of this angle so that $\overline{BC}$ is 3 inches long. Draw $\overline{CD}$. What is the measure of $\overline{CD}$? What are the measures of the angles at C and D? Your figure should be similar to Figure 4-4c.

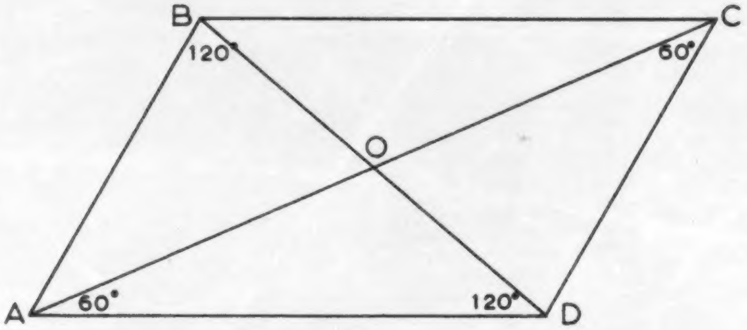


Figure 4-4c

Consider $\overline{AB}$ as a transversal cutting $\overline{AD}$ and $\overline{BC}$. In view of the sum of the measures of angle $\angle$ DAB and $\angle$ ABC, what is the relationship of $\overline{AD}$ and $\overline{BC}$? What can you say with regard to the relationship of $\overline{AB}$ and $\overline{CD}$? Are they parallel? What kind of quadrilateral is your polygon?

Now draw diagonals $\overline{BD}$ and $\overline{AC}$ intersecting at O. Measure $\overline{OB}$ and $\overline{OD}$. Do they seem to be equal? What about the measures of $\overline{OA}$ and $\overline{OC}$? Would it seem, then, that the diagonals of a parallelogram bisect each other? It is true that they do. Let us write the proof. Of course, we shall have to use parallel line facts which were not proved to be true. You were told, however, that they were true and they seem to be true by measurement. Always keep in mind that measurements are only approximate.

Consider the triangles AOD and BOC of Figure 4-4c. Now $\overline{AC}$ is a transversal cutting parallels $\overline{AD}$ and $\overline{BC}$. In view of your work with parallel lines m ($\angle$ OAD) = m ($\angle$ OCB). In a similar way you should see that m ($\angle$ ODA) = m ($\angle$ OBC). Since $\overline{AD}$ and $\overline{BC}$ are opposite sides of a parallelogram they are congruent. We now have the following information:

$$\angle OAD \cong \angle OCB$$
$$\angle ODA \cong \angle OBC$$
$$\overline{AD} \cong \overline{BC}.$$

You can now see that we have two triangles, AOD and BOC, which have two corresponding angles and the included sides congruent. By Property 2, these triangles are congruent. Since they are congruent, then their corresponding sides are congruent. That is:

$$\overline{OA} \cong \overline{OC}$$

and

$$\overline{OB} \cong \overline{OD}.$$

We have thus shown that the diagonals bisect each other. You will do more work of this kind when you get to 10th Grade Geometry. This method of proof is called the deductive method.

## Rectangles and Squares

There are 2 types of polygons of 4 sides which are special cases of a parallelogram. These 2 types of polygons are not only interesting, but useful.

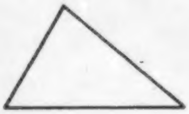
1. Rectangle. A rectangle is a parallelogram in which each angle measures 90. Using your tools of geometry, carefully construct a parallelogram having two opposite sides each 4 inches long, two opposites sides each 6 inches long, and all angles measure 90. Note that the rectangle has all of the properties

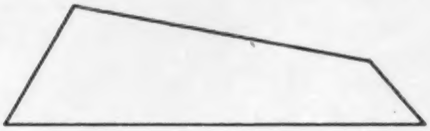
which apply to the parallelogram.

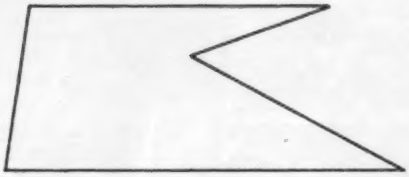
2. Square. Repeat the above construction, but this time let all the sides be 4 inches in length. Again you have a special kind of parallelogram having all sides congruent and all angles measure 90. The square will be important in the next section.

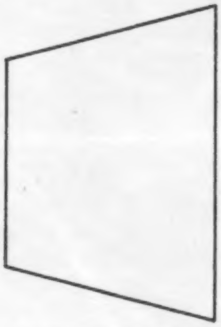
## Exercises 4-4

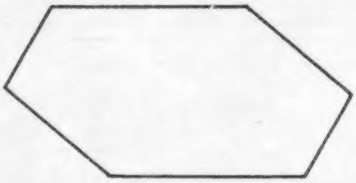
1. Which of the following figures are quadrilaterals?

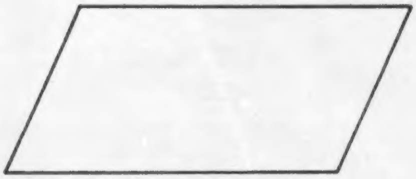












2. Which of the figures in exercise 1 appear to be parallelograms?

3. Given a line segment $\overline{MN}$. Now, using your straightedge and compass, do the following: Construct the perpendicular bisector of $\overline{MN}$. Select any point R on the perpendicular bisector, which is not on $\overline{MN}$, and construct a perpendicular to it at point R. Let this perpendicular be $\overline{RT}$. Are the segments $\overline{MN}$ and $\overline{RT}$ parallel? Why?

4. Using your straightedge and protractor, draw a parallelogram having all sides 4 inches long and its angles equal in measure to 80 and 100. Is this a square? Why?

5. Given a line segment $\overline{XY}$, 4 inches long. Using your straightedge and compass, construct a square with $\overline{XY}$ as one of its sides. Keep in mind that the intersecting angles of perpendicular lines measure 90.

6. Given a line segment $\overline{AB}$. Using your straightedge and compass, make the following construction. At point B construct the perpendicular to $\overline{AB}$. From any point C on this perpendicular draw $\overline{AC}$. What polygon do you have? Now construct a square on each sides of the triangle. You will have three squares, such that $\overline{AB}$ is a side of one square, $\overline{BC}$ is a side of a second square, and $\overline{AC}$ is a side of the third square.

7. Can a square be classified as a rectangle? Explain your answer.

8. Can a rectangle be classified as a square? Explain your answer.

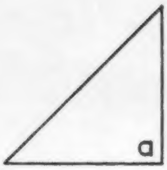
9. Do the diagonals of a square bisect each other. Explain your answer.

10. Return to your figure of exercise 5. Draw the diagonals. What relationship do you think exists among the intersecting angles of the diagonals? Maybe your protractor will aid you in answering the question. Keep in mind that a general statement cannot be made, based upon measurements. By use of congruent triangles you could prove your answer. Give it a try.

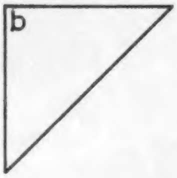
## 4-5. Right Triangle

Sometimes it is convenient to name triangles according to measures of the angles. Consider the following three sets of triangles:

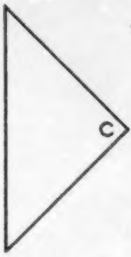
(a)



$m(\angle a)$ is 90

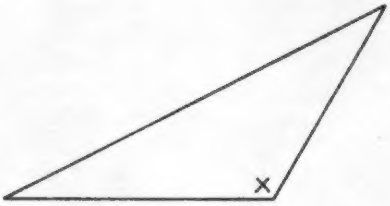


$m(\angle b)$ is 90

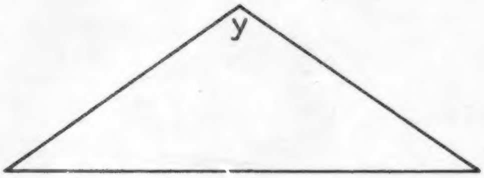


$m(\angle c)$ is 90

(b)

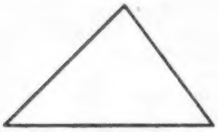


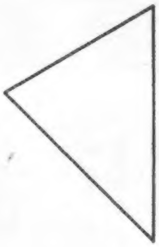
$m(\angle x)$ is 120

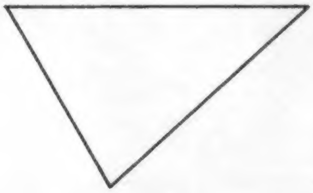


$m(\angle y)$ is 110

(c)







The triangles in set (a) each contain an angle with measure of $90^{\circ}$. Triangles having this property are called right triangles. The triangles in set (b) each contain an obtuse angle (an angle with measure greater than 90). Triangles having this property are called obtuse triangles. Triangles in set (c) contain only acute angles (angles with measure less than 90). Triangles having this property are called acute triangles.

You are now almost ready to study a very important and useful property of right triangles. Let us study the square further before taking up this property. Consider the following figure:

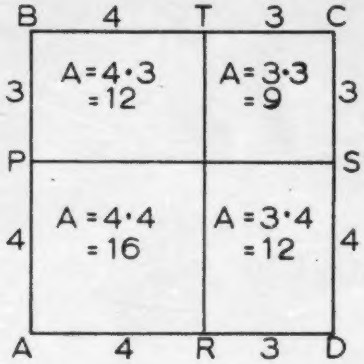


Figure 4-5a

In this figure we have the square ABCD, with each side having a measure of 7 units. You remember, of course, that the area of a square is equal in measure to the square of one side: that is, $A = s^2$, where A stands for area and s represents the length of one side. The area of ABCD is then: $A = 7 \cdot 7 = 49$ square units. Looking again at Figure 4-5a you will notice that it has been divided into 4 parts. These parts consist of two squares and two rectangles. You will further notice that the area for each part is indicated. How does the sum of the area of the 4 parts compare with the area of the square ABCD?

Let us consider the same square divided into parts in a different way as follows:

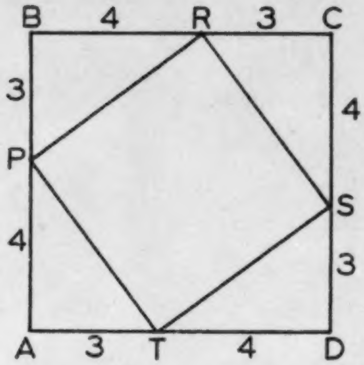


Figure 4-5b

Again, the area of square ABCD is equal to 49 square units in measure. This time the square is divided into 4 right triangles and the quadrilateral PRST. Why are the 4 triangles right triangles? You remember that the area of a triangle is equal in measure to $\frac{1}{2}$ of the product of the base and the altitude. Since the 4 triangles are all right triangles, you may consider the base of each triangle to be 4 and the altitude to be 3. The area of each triangle is then: $A = \frac{1}{2}(4 \cdot 3) = 6$ square units. The total area of the 4 triangles is 24 square units. Can you say the 4 triangles are congruent? Why?

What is the area of the square ABCD? You, of course, get 49 square units. Now, since the area of the square ABCD is 49 square units and the total area of the 4 triangles is 24 square units, the area of the quadrilateral PRST is 25 square units. What special kind of quadrilateral do you think PRST might be? Since the 4 triangles are congruent, then $\overline{PR}$, $\overline{RS}$, $\overline{ST}$, and $\overline{TP}$ are congruent by Property 4. Hence, PRST is at least a parallelogram, since its opposite sides are congruent. Also, since the triangles are congruent, $\angle$ BPR $\cong$ $\angle$ CRS, by Property 4. Now let us consider the following:

1. You know that m ($\angle$ BRP + m ($\angle$PRS) + m ($\angle$ SRC) is 180. Why?
2. You also know that m ($\angle$ BPR) + m ($\angle$ PBR) + m ($\angle$ BRP) is 180. Why?
3. Now m ($\angle$ PBR) is 90. Why? What then must be m ($\angle$ BPR) + m ($\angle$ BRP)? With some thought you will realize it is 90, because 180 - 90 = 90.
4. Since $\angle$ BPR $\cong$ $\angle$ CRS, by Property 4, and m ( $\angle$ BPR) + m ($\angle$ BRP) is 90 from item 3 above, you should see that m ($\angle$ BRP) + m ($\angle$ CRS) is 90.
5. Now since m ($\angle$ BRP) + m ($\angle$ PRS) + m ($\angle$ SRC) is 180, from item 1, and m ($\angle$ BRP) + m ($\angle$ CRS) is 90, from item 4, you should see that m ($\angle$ PRS) is 90.

You can see that quadrilateral PRST is a square. Let us summarize what we have learned.

1. PRST is a quadrilateral because it is a polygon of 4 sides.
2. PRST is a parallelogram because its opposite sides are congruent. Furthermore, we know that all of its sides have equal measure. We know this because the 4 triangles are congruent.
3. We have shown that m ($\angle$ PRS) is 90. In a similar manner we can show that $\angle$ RST, $\angle$ STP, and $\angle$ TPR measure 90.

Hence, we have a quadrilateral whose sides and angles are all equal. This kind of polygon, as you know, is a square.

Since PRST is a square, the measure of whose area is 25 square units, what do you suppose is the measure on one side? Since the area of a square whose side measures 5 inches is 5 · 5 = 25 square inches, we know then, the sides of PRST each measure 5 units.

Let us now consider the right triangle PBR in Figure 4-5b. Suppose we construct a square on each side of the triangle as shown in Figure 4-5c.

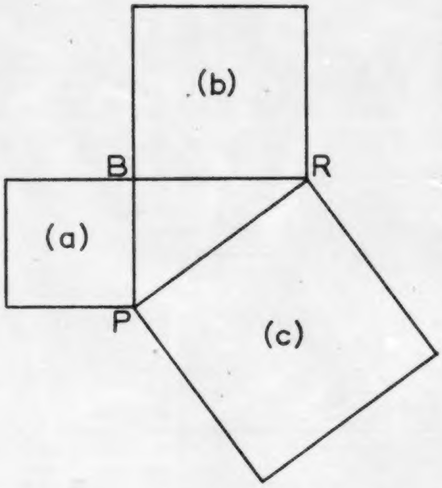


Figure 4-5c

What are the areas of these three squares? You know that the area of square (a) is 9, of square (b) is 16, and of square (c) is 25. What do we get when we add the areas of square (a) and square (b)? You see that the sum of the areas of (a) and (b) is 25, the same area as square (c).

The side opposite the right angle of a right triangle is called the hypotenuse. In this special case we have seen that the measure of the square on the hypotenuse of a right triangle is equal to the sum of the measures of the squares on the other two sides. In this work you have been introduced to one of the most important and one of the most beautiful theorems in all mathematics. Of course, we have not proved the theorem in general, but it is true. The theorem may be stated as follows:

> In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Given the right triangle ABC as shown in Figure 4-5d, where m (∠ ACB) is 90.

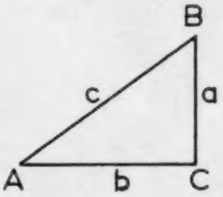


Figure 4-5d

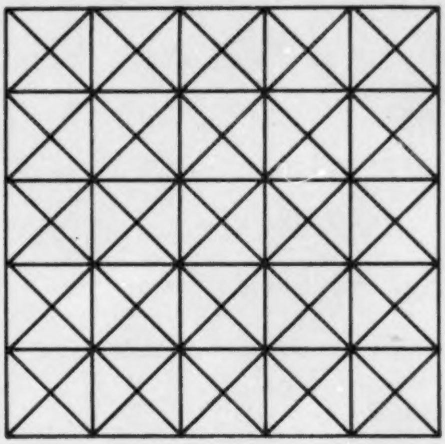
Let us name $\overline{BC}$, a; $\overline{AC}$, b; and $\overline{AB}$, c. You may now state the theorem in the following form:

$$c^2 = a^2 + b^2.$$

Special cases of this theorem were known by the Babylonians and the Egyptians at least some 2000 to 3000 years before Christ. It remained, however, for some mathematician of the Pythagorean School to prove the theorem for the general case. For this reason the theorem is know as the Pythagorean Theorem in honor of Pythagoras, after whom the school was named. (When and where did Pythagoras live?)

## Exercises 4-5

1. It has been said that Pythagoras noticed this property of the



right triangle by looking at a mosaic like the one above. Does this help you to see the property better? Suppose the mosaic is constructed from small squares which are congruent. What can you then say about the small triangles formed by drawing the diagonals of the squares? Now compare the number of triangles in the square of the hypotenuse with the sum of the triangles in the squares of the other two sides.

2. Using your straightedge and protractor draw the following triangles:

   (a) Obtuse triangle

   (b) Acute triangle

   (c) Right triangle

3. Carefully reproduce Figure 4-5a and call this your Figure 1. Also carefully reproduce Figure 4-5b and call it your Figure 2. Use an inch as your unit. Now cut out your Figure 1 and cut it into the two squares and the two rectangles as

indicated in the figure. Next, divide the two rectangles into two equal triangles each. Will these triangles fit on top of the triangles in your Figure 2? What can you now say with respect to the sum of the areas of the two squares in your Figure 1, and the area of the quadrilateral PRST in your Figure 2? Now fit the two squares of Figure 1 on the sides of triangle BPR in an arrangement similar to Figure 4-5c. What theorem have you now demonstrated?

4. Show for the following numbers that the square of the first is the sum of the squares of the others in each set of 3:

| | |
|---|---|
| (a) 5, 4, 3 | (c) 25, 7, 24 |
| (b) 13, 5, 12 | (d) 20, 16, 12 |

5. Draw or construct triangles with the sides of length (in centimeters) given in the parts (a) and (b) of exercise 3. Use your protractor to show that these triangles are right triangles.

6. Draw right triangles, the lengths of whose shorter sides (in centimeters), are:

   (a) 1 and 2 (b) 4 and 5 (c) 2 and 3.

   Measure with a ruler to the nearest one-tenth of a centimeter, if possible, the lengths of the hypotenuses of these triangles.

7. Use the right triangle principle to find the squares of the lengths of the hypotenuses of the triangles in exercise 6.

8. In using unsigned numbers, we say that the square root of 25 is 5 and the square root of 16 is 4. We would say that the square root of 5 is a number such that its square is 5. We know that there is no fraction or whole number of which this

is true. We can, however, find a number for which this is true. The approximate values of square roots of some of the whole numbers are given in the table of square roots. Use the table (See last page of this Unit.) of square roots to find approximate values of the square roots of the following:

(a) 5 (b) 41 (c) 13.

9. Compare the results in exercises 6 and 8. Explain, using the property of right triangles, why we might expect the result of exercises 6 (a) and 8 (a) to be about the same. Do the same for the parts (b) and (c) of exercises 6 and 8.

10. Use the property of right triangles to find the lengths of the hypotenuses of right triangles with sides given of the following lengths:

(a) sides of length 3 units and 5 units

The square of the length of the hypotenuse is

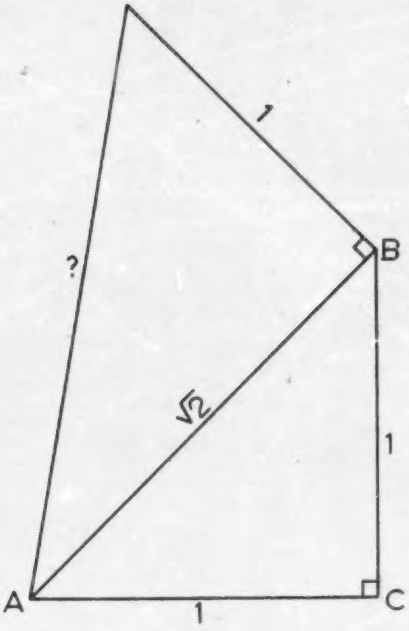
$$3^2 + 5^2 = 34.$$

The length of the hypotenuse is the square root of 34. From the table we find that this is 5.8, correct to the nearest one-tenth.

(b) 5 and 6 (c) 3 and 9 (d) 1 and 3.

*11. Draw a square whose sides are of length 1 unit. What is the length of the diagonal? Check by measurement. Now draw a right triangle with the sides 1 unit long. What is the length of the hypotenuse?

*12. Now draw a right triangle of sides "square root of 2" and 1 unit in length as shown in the figure. In the figure the length of $\overline{AB}$ is the square root of 2. What is the length of the hypotenuse of this new triangle?


1
?
B
$\sqrt{2}$
1
A
1
C

# TABLE

## SQUARES AND SQUARE ROOTS OF NUMBERS

| No. | Squares | Square roots | No. | Squares | Square roots |
|---|---|---|---|---|---|
| 1 | 1 | 1.000 | 36 | 1,296 | 6.000 |
| 2 | 4 | 1.414 | 37 | 1,369 | 6.083 |
| 3 | 9 | 1.732 | 38 | 1,444 | 6.164 |
| 4 | 16 | 2.000 | 39 | 1,521 | 6.245 |
| 5 | 25 | 2.236 | 40 | 1,600 | 6.325 |
| 6 | 36 | 2.449 | 41 | 1,681 | 6.403 |
| 7 | 49 | 2.646 | 42 | 1,764 | 6.481 |
| 8 | 64 | 2.828 | 43 | 1,849 | 6.557 |
| 9 | 81 | 3.000 | 44 | 1,936 | 6.633 |
| 10 | 100 | 3.162 | 45 | 2,025 | 6.708 |
| 11 | 121 | 3.317 | 46 | 2,116 | 6.782 |
| 12 | 144 | 3.464 | 47 | 2,209 | 6.856 |
| 13 | 169 | 3.606 | 48 | 2,304 | 6.928 |
| 14 | 196 | 3.742 | 49 | 2,401 | 7.000 |
| 15 | 225 | 3.873 | 50 | 2,500 | 7.071 |
| 16 | 256 | 4.000 | 51 | 2,601 | 7.141 |
| 17 | 289 | 4.123 | 52 | 2,704 | 7.211 |
| 18 | 324 | 4.243 | 53 | 2,809 | 7.280 |
| 19 | 361 | 4.359 | 54 | 2,916 | 7.348 |
| 20 | 400 | 4.472 | 55 | 3,025 | 7.416 |
| 21 | 441 | 4.583 | 56 | 3,136 | 7.483 |
| 22 | 484 | 4.690 | 57 | 3,249 | 7.550 |
| 23 | 529 | 4.796 | 58 | 3,364 | 7.616 |
| 24 | 576 | 4.899 | 59 | 3,481 | 7.681 |
| 25 | 625 | 5.000 | 60 | 3,600 | 7.746 |
| 26 | 676 | 5.099 | 61 | 3,721 | 7.810 |
| 27 | 729 | 5.196 | 62 | 3,844 | 7.874 |
| 28 | 784 | 5.292 | 63 | 3,969 | 7.937 |
| 29 | 841 | 5.385 | 64 | 4,096 | 8.000 |
| 30 | 900 | 5.477 | 65 | 4,225 | 8.062 |
| 31 | 961 | 5.568 | 66 | 4,356 | 8.124 |
| 32 | 1,024 | 5.657 | 67 | 4,489 | 8.185 |
| 33 | 1,089 | 5.745 | 68 | 4,624 | 8.246 |
| 34 | 1,156 | 5.831 | 69 | 4,761 | 8.307 |
| 35 | 1,225 | 5.916 | 70 | 4,900 | 8.367 |

| No. | Squares | Square roots |
|---|---|---|
| 71 | 5,041 | 8.426 |
| 72 | 5,184 | 8.485 |
| 73 | 5,329 | 8.544 |
| 74 | 5,476 | 8.602 |
| 75 | 5,625 | 8.660 |
| 76 | 5,776 | 8.718 |
| 77 | 5,929 | 8.775 |
| 78 | 6,084 | 8.832 |
| 79 | 6,241 | 8.888 |
| 80 | 6,400 | 8.944 |
| 81 | 6,561 | 9.000 |
| 82 | 6,724 | 9.055 |
| 83 | 6,889 | 9.110 |
| 84 | 7,056 | 9.165 |
| 85 | 7,225 | 9.220 |
| 86 | 7,396 | 9.274 |
| 87 | 7,569 | 9.327 |
| 88 | 7,744 | 9.381 |
| 89 | 7,921 | 9.434 |
| 90 | 8,100 | 9,487 |
| 91 | 8,281 | 9.539 |
| 92 | 8,464 | 9.592 |
| 93 | 8,649 | 9.644 |
| 94 | 8,836 | 9.695 |
| 95 | 9,025 | 9.747 |
| 96 | 9,216 | 9.798 |
| 97 | 9,409 | 9.849 |
| 98 | 9,604 | 9.899 |
| 99 | 9,801 | 9.950 |
| 100 | 10,000 | 10.000 |